Circles

[Center C(h, k) and Radius r]

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The **circle** is an important application of the distance formula.

Definition: A **circle** is the set of all points **P(x , y)** that are a fixed distance **r**, called the **radius**, from a fixed point **C(h ,k)**, called the **center.**

Circle Equation: The **equation of the circle** with **radius** $\mathbf{r}$ and **center** $\mathbf{C(h,k)}$, is given by

$$\sqrt{(\mathbf{x}-\mathbf{h})^2+(\mathbf{y}-\mathbf{k})^2}=\mathbf{r} \ ; \ (\mathbf{x}-\mathbf{h})^2+(\mathbf{y}-\mathbf{k})^2=\mathbf{r}^2$$

where **P(x , y)** represents a point on the circle.

Its **extreme points,** points on the graph that have a maximum/minimum x or y values, are

$$(\mathbf{h}-\mathbf{r},\mathbf{k}) \ ; \ (\mathbf{h}+\mathbf{r},\mathbf{k})$$
$$(\mathbf{h},\mathbf{k}-\mathbf{r}) \ ; \ (\mathbf{h},\mathbf{k}+\mathbf{r})$$

The **domain,** the projection of the graph onto the x-axis, is $[\mathbf{h}-\mathbf{r},\mathbf{h}+\mathbf{r}]_{\mathbf{x}}$. These are the *allowable* x values.

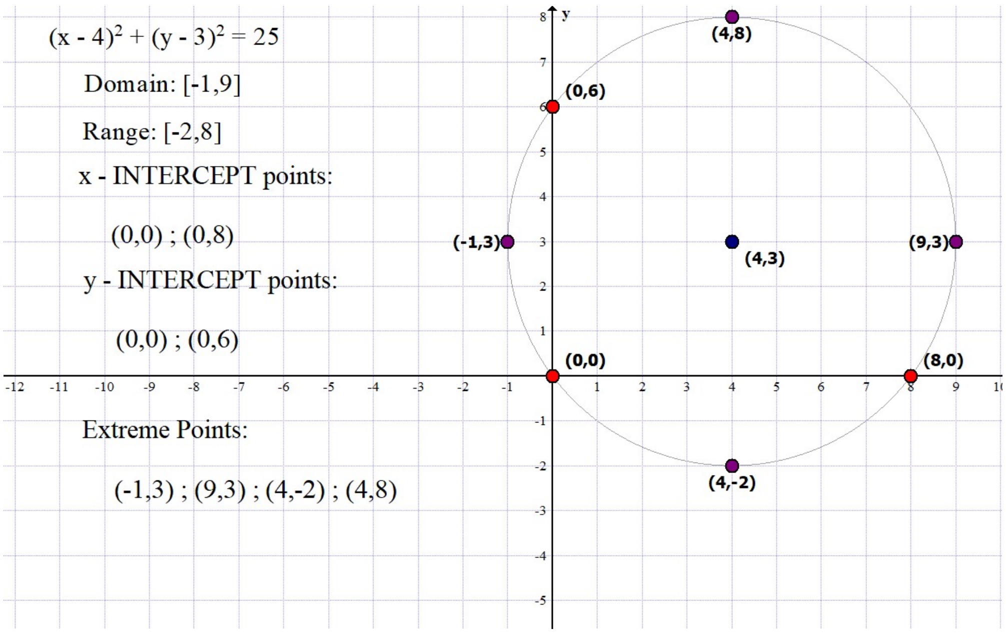
The **range**, the projection of the graph onto the y-axis, is $[\mathbf{k}-\mathbf{r},\mathbf{k}+\mathbf{r}]_{\mathbf{y}}$ These are the *allowable* y values.

The x coordinates of the **x-intercept points**, if any, are solutions of $(\mathbf{x}-\mathbf{h})^2+(0-\mathbf{k})^2=\mathbf{r}^2$ **Note:** y = 0

The y coordinates of the **y-intercept points**, if any, are solutions of $(0-\mathbf{h})^2+(\mathbf{y}-\mathbf{k})^2=\mathbf{r}^2$ **Note:** x = 0

Note: These are the points where the graph of the circle intersects with either the x-axis or y-axis.

The following graph shows an example using these definitions:



Example 01: Given $C(h,k) = C(1,-2)$ and radius $r = 3$, find the following:

a. The **equation** of the circle:

Step	Equation	Reason
0	$(x-h)^2 + (y-k)^2 = r^2$	
1	$(x-[1])^2 + (y-[-2])^2 = [3]^2$	Be careful with the minus signs
2	$(x-1)^2 + (y+2)^2 = 3^2$	
3	$x^2 - 2x + 1 + y^2 + 4y + 4 = 9$	
4	$x^2 + y^2 - 2x + 4y - 4 = 0$	Quadratic Form

b. The **extreme points**:

Step	Extreme Points	Reason
0	$(\mathbf{h}-\mathbf{r},\mathbf{k})$; $(\mathbf{h}+\mathbf{r},\mathbf{k})$	
1	Points: $(-2,-2)$; $(4,-2)$	
0	$(\mathbf{h},\mathbf{k}-\mathbf{r})$; $(\mathbf{h},\mathbf{k}+\mathbf{r})$	
1	Points: $(1,-5)$; $(1,1)$	

Note: The center $\mathbf{C}(1,-2)$ is the mid-point of the extreme points:

1. $(-2,-2)$; $(4,-2)$
2. $(1,-5)$; $(1,1)$

c. The **domain**: $[-2,4]$

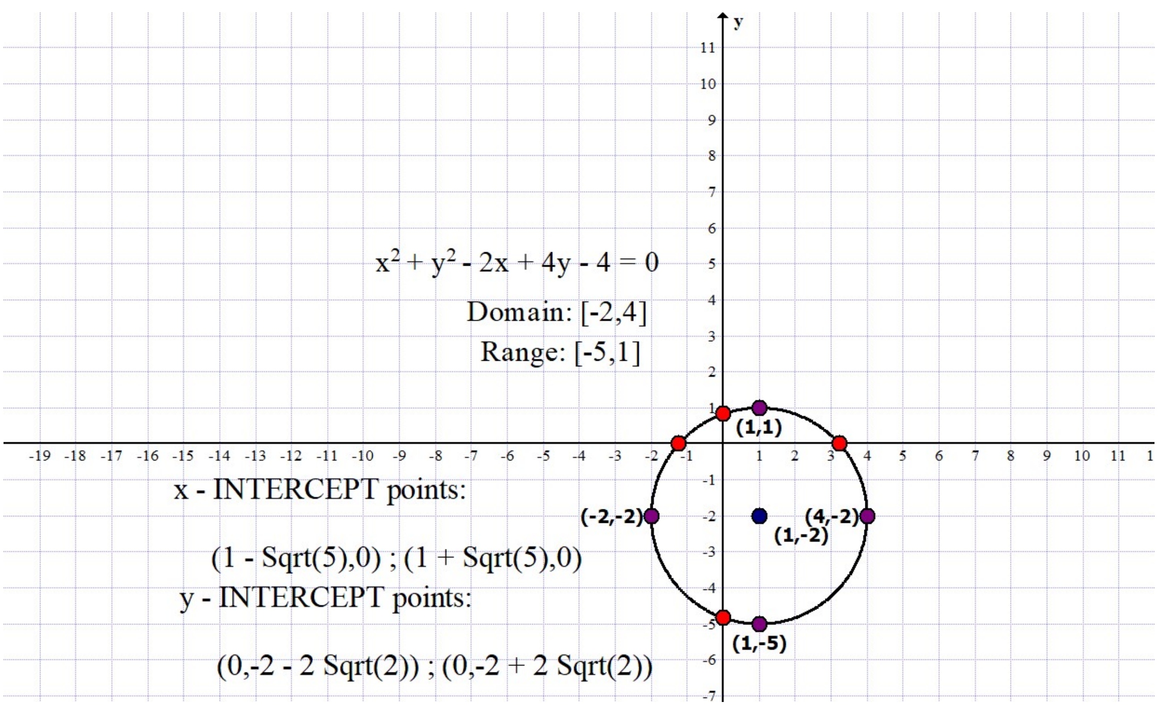
d. The **range**: $[-5,1]$

e. The **x-intercept points**: Set $\mathbf{y}=0$ and solve for x

Step	x-intercept points	Reason
0	$\mathbf{x}^2-2\mathbf{x}-4=0$	Solve
1	$\mathbf{x}=\dfrac{-\mathbf{b}\pm\sqrt{\mathbf{b}^2-4\mathbf{ac}}}{2\mathbf{a}}$; $\mathbf{a}=1$; $\mathbf{b}=-2$; $\mathbf{c}=-4$	
2	$\mathbf{x}=\dfrac{-[-2]\pm\sqrt{[-2]^2-4[1][-4]}}{2[1]}$	
3	$\mathbf{x}=\dfrac{2\pm\sqrt{4+16}}{2}$	
4	$\mathbf{x}=\dfrac{2\pm\sqrt{20}}{2}=\dfrac{2\pm2\sqrt{5}}{2}$	
5	$\mathbf{x}=1\pm\sqrt{5}$	
6	$\mathbf{x}=1-\sqrt{5}\approx-1.236$ $\Vert$ $\mathbf{x}=1+\sqrt{5}\approx3.236$	
7	Points: $(1-\sqrt{5},0)$; $(1+\sqrt{5},0)$ (-1.236, 0) ; (3.236, 0)	

The **y-intercept points**: Set $\mathbf{x}=0$ and solve for y

Step	y-intercept points	Reason
0	$\mathbf{y}^2+4\mathbf{y}-4=0$	Solve
1	$\mathbf{y}=\dfrac{-\mathbf{b}\pm\sqrt{\mathbf{b}^2-4\mathbf{ac}}}{2\mathbf{a}}$; $\mathbf{a}=1$; $\mathbf{b}=4$; $\mathbf{c}=-4$	
2	$\mathbf{y}=\dfrac{-[4]\pm\sqrt{[4]^2-4[1][-4]}}{2[1]}$	
3	$\mathbf{y}=\dfrac{-4\pm\sqrt{16+16}}{2[1]}$	
4	$\mathbf{y}=\dfrac{-4\pm\sqrt{32}}{2}=\dfrac{-4\pm4\sqrt{2}}{2}$	
5	$\mathbf{y}=-2\pm2\sqrt{2}$	
6	$\mathbf{y}=-2-2\sqrt{2}\approx-4.83$; $\mathbf{y}=-2+2\sqrt{2}\approx+0.83$	
7	Points: $(0,-2-2\sqrt{2})$; $\left(0,-2+2\sqrt{2}\right)$ (0, -4.828) ; (0, 828)	



Note: Always draw the graph and make sure that *all* points are where they are supposed to be. If NOT, "Fix it"!

Circles, ellipses, parabolas, and hyperbolas have the following form:

General Quadratic Equation in x & y:

$$\mathbf{A}\mathbf{x}^2+\mathbf{B}\mathbf{x}\mathbf{y}+\mathbf{C}\mathbf{y}^2+\mathbf{D}\mathbf{x}+\mathbf{E}\mathbf{y}+\mathbf{F}=0$$

This will be a circle if $\mathbf{A}=\mathbf{C}\neq 0 \quad (\mathbf{B}=0)$. We must *complete the square* twice, once for "x" and once for "y", to determine the Center & Radius.

Example 02: Given the circle defined by $\mathbf{x}^2+\mathbf{y}^2+6\mathbf{x}-10\mathbf{y}+9=0$, find the following:

a. The **center** $\mathbf{C}(\mathbf{h},\mathbf{k})$ and **radius** $\mathbf{r}$:

Step	Equation	Reason
0	$\mathbf{x}^2+\mathbf{y}^2+6\mathbf{x}-10\mathbf{y}+9=0$	A = 1 = C B = 0
1	$\mathbf{x}^2+6\mathbf{x}+[9]+\mathbf{y}^2-10\mathbf{y}+[25]=-9+[9]+[25]$	$\left[\frac{1}{2}(6)\right]^2=9$ $\left[\frac{1}{2}(-10)\right]^2=25$
2	$(\mathbf{x}+3)^2+(\mathbf{y}-5)^2=5^2$	
3	Center: $\mathbf{C}(-3,5)$; Radius: $\mathbf{r}=5$	

b. The **extreme points**:

Step	Extreme Points	Reason
0	$(\mathbf{h}-\mathbf{r},\mathbf{k})$; $(\mathbf{h}+\mathbf{r},\mathbf{k})$	
1	Points: $(-8,5)$; $(2,5)$	
0	$(\mathbf{h},\mathbf{k}-\mathbf{r})$; $(\mathbf{h},\mathbf{k}+\mathbf{r})$	
1	Points: $(-3,0)$; $(-3,10)$	

c. The **domain**: $[-8,2]$

d. The **range**: $[0,10]$

e. The **x-intercept points**: Set $\mathbf{y} = 0$

Step	x-intercept points	Reason
0	$\mathbf{x}^2 + 6\mathbf{x} + 9 = 0$	Solve
1	$(\mathbf{x}+3)^2 = 0$	
2	$\mathbf{x}+3=0$ $\mathbf{x}=-3$	
3	Point: $(-3,0)$	

f. The **y-intercept points**: Set $\mathbf{x} = 0$

Step	y-intercept points	Reason
0	$\mathbf{y}^2 - 10\mathbf{y} + 9 = 0$	Solve
1	$(\mathbf{y}-1)(\mathbf{y}-9) = 0$	
2	$\mathbf{y}-1=0$ ‖ $\mathbf{y}-9=0$ $\mathbf{y}=1$ ‖ $\mathbf{y}=9$	
3	Points: $(0,1)$; $(0,9)$	

Graph: