Algebra of FUNctions

Sum, Difference, Product, Quotient, and Composition

$\mathbf{f+g}$ $\mathbf{f-g}$ $\mathbf{f \cdot g}$ $\mathbf{f/g}$ $\mathbf{f \circ g}$

of
FUNctions

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We now create *new* functions from *old* (given) functions. Given two (2) functions f and g, we form five (5) new functions. It makes sense that we can form the Sum, Difference, Product, and Quotient of two (2) functions since we have done this with numbers and functions are just collections of ordered pairs of numbers. The Composition, although very important, is more complicated so we consider it last. Because finding their properties and graphs frequently requires techniques outside of those studied at this time, we just concentrate on finding their formulas and domains.

NEW FUNctions from OLD (GIVEN) FUNctions:

For each "new function", we specify the following:

1. Name
2. Domain
3. Correspondence

Old: $\mathbf{f(x)\ ;\ g(x)}$

a. **New** – **Sum**: $\mathbf{f+g}$
   $\mathbf{Dom\ (f+g) = Dom\ f \cap Dom\ g}$
   $\mathbf{(f+g)(x) = f(x) + g(x)}$

b. **New** - **Difference**: $\mathbf{f-g\ ;\ g-f}$
   $\mathbf{Dom\ (f-g) = Dom\ f \cap Dom\ g}$
   $\mathbf{(f-g)(x) = f(x) - g(x)}$

c. **New** - **Product**: $\mathbf{fg}$
   $\mathbf{Dom\ (fg) = Dom\ f \cap Dom\ g}$
   $\mathbf{(fg)(x) = f(x)\ g(x)}$

d. **New** - **Quotient**: $\frac{f}{g}$; $\frac{g}{f}$

$$\textbf{Dom }\left(\frac{f}{g}\right) = \textbf{Dom f} \cap \{x \in \textbf{Dom g} \mid g(x) \neq 0\}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

e. **New** - **Composition**: $f \circ g$; $g \circ f$

$$\textbf{Dom }(f \circ g) = \{x \in \textbf{Dom g} \mid g(x) \in \textbf{Dom f}\}$$

$$(f \circ g)(x) = f(g(x))$$

Again, since we studied sums, differences, products, and quotients of numbers many years ago, and functions are *just* collections of numbers, we study the new functions a – d first.

Example 01: Consider the functions $f(x)$ and $g(x)$ defined in the following tables:

x	f(x)
-2	4
-1	2
0	1
3	5
5	4

x	g(x)
-3	-1
-1	5
0	3
2	-5
5	0

Find the Sum, Difference, Product, and Quotient of $f(x)$ and $g(x)$.

Solution:

First, $\textbf{Dom f} \cap \textbf{Dom g} = \{-1, 0, 5\}$. Using the definitions above, we obtain

x	$(f+g)(x)$	$(f-g)(x)$	$(fg)(x)$	$\left(\frac{f}{g}\right)(x)$
-1	7	-3	10	2/5
0	4	-2	3	1/3
5	4	4	0	Trash!

Example 02: Consider the functions $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ defined below:

$$\mathbf{f}(\mathbf{x}) = \sqrt{3\mathbf{x} - 2} \;\; ; \; \mathbf{g}(\mathbf{x}) = \sqrt{4 - 5\mathbf{x}}$$

Find the Sum, Difference, Product, and Quotient of $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$.

Solution:

First, note that

$$\textbf{Dom f} = \left[\frac{2}{3}, +\infty\right) \; ; \; \textbf{Dom g} = \left(-\infty, \frac{4}{5}\right]:$$

___________[/////////////////////////// **Dom f**

$\frac{2}{3}$

///////////////////////////]__________ **Dom g**

$\frac{4}{5}$

___________[/////////////]_________ **Dom f ∩ Dom g**

$\frac{2}{3}$ $\frac{4}{5}$

since $3\mathbf{x} - 2 \ge 0 \Rightarrow 3\mathbf{x} \ge 2 \Rightarrow \mathbf{x} \ge \frac{2}{3}$ and since $4 - 5\mathbf{x} \ge 0 \Rightarrow 4 \ge 5\mathbf{x} \Rightarrow \frac{4}{5} \ge \mathbf{x} \Rightarrow \mathbf{x} \le \frac{4}{5}$.
Using our definitions, we have

Sum:

$$\textbf{Dom}\,(\mathbf{f} + \mathbf{g}) = \left[\frac{2}{3}, \frac{4}{5}\right]$$

$(\mathbf{f} + \mathbf{g})(\mathbf{x}) \equiv \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) = \sqrt{3\mathbf{x} - 2} + \sqrt{4 - 5\mathbf{x}}$; does NOT simplify ...

Difference:

$$\mathbf{Dom}\left(\mathbf{f}-\mathbf{g}\right)=\left[\frac{2}{3},\frac{4}{5}\right]$$

$$\left(\mathbf{f}-\mathbf{g}\right)(\mathbf{x})\equiv\mathbf{f}(\mathbf{x})-\mathbf{g}(\mathbf{x})=\sqrt{3\mathbf{x}-2}-\sqrt{4-5\mathbf{x}}$$; does NOT simplify...

Product:

$$\mathbf{Dom}\left(\mathbf{f}\,\mathbf{g}\right)=\left[\frac{2}{3},\frac{4}{5}\right]$$

$$\left(\mathbf{f}\,\mathbf{g}\right)(\mathbf{x})\equiv\mathbf{f}(\mathbf{x})\mathbf{g}(\mathbf{x})=\sqrt{3\mathbf{x}-2}\,\sqrt{4-5\mathbf{x}}$$

$$=\sqrt{\left(3\mathbf{x}-2\right)\left(4-5\mathbf{x}\right)}=\sqrt{-15x^2+22\mathbf{x}-8}$$

Quotient:

$$\mathbf{Dom}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)=\left[\frac{2}{3},\frac{4}{5}\right)$$

$$\left(\frac{\mathbf{f}}{\mathbf{g}}\right)(\mathbf{x})\equiv\frac{\mathbf{f}(\mathbf{x})}{\mathbf{g}(\mathbf{x})}=\frac{\sqrt{3\mathbf{x}-2}}{\sqrt{4-5\mathbf{x}}}=\sqrt{\frac{3\mathbf{x}-2}{4-5\mathbf{x}}}$$

Example 03: Consider the functions $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ defined below:

$$\mathbf{f}(\mathbf{x})=\frac{\mathbf{x}}{\mathbf{x}-1}\quad;\quad \mathbf{g}(\mathbf{x})=\frac{\mathbf{x}+1}{\mathbf{x}}$$

Find the Sum, Difference, Product, and Quotient of $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$.

Solution:

First, note that

$$\textbf{Dom f}=\mathbb{R}\setminus\{1\}\ ;\ \textbf{Dom g}=\mathbb{R}\setminus\{0\}$$

/ ∘ / / / / / / / / / / / / / / / / **Dom f**

1

/ / / / / / / / / / / / / / / / ∘ / **Dom g**

0

/ / / / / / / / / / / / / / / / ∘ / / / / / / / / ∘ / / / / / / / / / / / / / / / / **Dom f** $\cap$ **Dom g**

0 1

We have

Sum:

$$\textbf{Dom}\,(\mathbf{f}+\mathbf{g})=\mathbb{R}\setminus\{0,1\}$$

$$(\mathbf{f}+\mathbf{g})(\mathbf{x})\equiv\mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x})=\frac{\mathbf{x}}{\mathbf{x}-1}+\frac{\mathbf{x}+1}{\mathbf{x}}$$

$$=\frac{\mathbf{x}^2+(\mathbf{x}^2-1)}{\mathbf{x}(\mathbf{x}-1)}=\frac{2\mathbf{x}^2-1}{\mathbf{x}(\mathbf{x}-1)}\text{ ; simplifies}$$

Difference:

$$\textbf{Dom}\,(\mathbf{f}-\mathbf{g})=\mathbb{R}\setminus\{0,1\}$$

$$(\mathbf{f}-\mathbf{g})(\mathbf{x})\equiv\mathbf{f}(\mathbf{x})-\mathbf{g}(\mathbf{x})=\frac{\mathbf{x}}{\mathbf{x}-1}-\frac{\mathbf{x}+1}{\mathbf{x}}$$

$$=\frac{\mathbf{x}^2-(\mathbf{x}^2-1)}{\mathbf{x}(\mathbf{x}-1)}=\frac{1}{\mathbf{x}(\mathbf{x}-1)}\text{ ; simplifies}$$

Product:

$$\mathbf{Dom}\left(\mathbf{f\,g}\right)=\mathbb{R}\setminus\{0,1\}$$

$$(\mathbf{f\,g})(\mathbf{x})\equiv\mathbf{f}(\mathbf{x})\mathbf{g}(\mathbf{x})=\frac{\mathbf{x}}{\mathbf{x}-1}\frac{\mathbf{x}+1}{\mathbf{x}}\underset{(\mathbf{x}\neq 0)}{=}\frac{\mathbf{x}+1}{\mathbf{x}-1}$$; simplifies

(Note: Do NOT simplify and then find the domain)

Quotient:

$$\mathbf{Dom}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)=\mathbb{R}\setminus\{-1,0,1\}$$; $\mathbf{g}(-1)=0\Rightarrow$ Trash!

$$\left(\frac{\mathbf{f}}{\mathbf{g}}\right)(\mathbf{x})\equiv\frac{\mathbf{f}(\mathbf{x})}{\mathbf{g}(\mathbf{x})}=\frac{\frac{\mathbf{x}}{\mathbf{x}-1}}{\frac{\mathbf{x}+1}{\mathbf{x}}}$$

$$\underset{(\mathbf{x}\neq 0)}{=}\frac{\mathbf{x}}{\mathbf{x}-1}\,\frac{\mathbf{x}}{\mathbf{x}+1}=\frac{\mathbf{x}^2}{\mathbf{x}^2-1}$$; simplifies

The composition of two (2) functions is *different* but extremely important! Please be careful when using its definition.

Example 04: Consider the functions $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ defined in the following tables:

x	g(x)
-3	3
-1	-1
0	4
1	-2
2	3
4	1

x	f(x)
-3	-1
-1	0
0	-1
1	3
3	-2
4	5

Find the Composition $\mathbf{f}(\mathbf{x})$ with $\mathbf{g}(\mathbf{x})$.

Solution:

We have

x	$(\mathbf{f} \circ \mathbf{g})(\mathbf{x})$
-3	-2
0	5
-1	0
2	-2
4	3

For example,

$$-3 \to \overbrace{3}^{\mathbf{g}(-3)} \to \overbrace{-2}^{\mathbf{f}(\mathbf{g}(-3))}$$

but

$$1 \to \overbrace{-2}^{\mathbf{g}(1)} \text{ is NOT in } \mathbf{Dom\,f}$$

Example 05: Consider the functions $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ defined below:

$$\mathbf{f}(\mathbf{x}) = 2\mathbf{x} - \mathbf{x}^2 \;\; ; \;\; \mathbf{g}(\mathbf{x}) = \mathbf{x}^2 + 2\mathbf{x} + 2$$

Find the Composition of $\mathbf{g}(\mathbf{x})$ with $\mathbf{f}(\mathbf{x})$ **and** $\mathbf{f}(\mathbf{x})$ with $\mathbf{g}(\mathbf{x})$

Solution:

First, note that

$$\textbf{Dom f} = \mathbb{R} \;\; ; \;\; \textbf{Dom g} = \mathbb{R}$$

Note:

1. $\mathbf{g}\left(\square\right) = \square^2 + 2\square + 2 \Rightarrow \mathbf{g}\left(\boxed{\mathbf{f}(\mathbf{x})}\right) = \boxed{\mathbf{f}(\mathbf{x})}^2 + 2\boxed{\mathbf{f}(\mathbf{x})} + 2$
2. $\mathbf{f}\left(\square\right) = 2\square - \square^2 \Rightarrow \mathbf{f}\left(\boxed{\mathbf{g}(\mathbf{x})}\right) = 2\boxed{\mathbf{g}(\mathbf{x})} - \boxed{\mathbf{g}(\mathbf{x})}^2$

We have

Composition: $\mathbf{g} \circ \mathbf{f}$

$$\textbf{Dom}\left(\mathbf{g} \circ \mathbf{f}\right) = \left\{ \mathbf{x} \in \overbrace{\textbf{Dom f}}^{\mathbb{R}} \;\middle|\; \mathbf{f}(\mathbf{x}) \in \overbrace{\textbf{Dom g}}^{\mathbb{R}} \right\} = \mathbb{R}$$

$$\begin{aligned}
\left(\mathbf{g} \circ \mathbf{f}\right)(\mathbf{x}) \equiv \mathbf{g}(\mathbf{f}(\mathbf{x})) &= \left[\mathbf{f}(\mathbf{x})\right]^2 + 2\left[\mathbf{f}(\mathbf{x})\right] + 2 \\
&= \left[2\mathbf{x} - \mathbf{x}^2\right]^2 + 2\left[2\mathbf{x} - \mathbf{x}^2\right] + 2 \\
&= 4\mathbf{x}^2 - 4\mathbf{x}^3 + \mathbf{x}^4 + 4\mathbf{x} - 2\mathbf{x}^2 + 2 \\
&= \mathbf{x}^4 - 4\mathbf{x}^3 + 2\mathbf{x}^2 + 4\mathbf{x} + 2
\end{aligned}$$

and

Composition: $\mathbf{f} \circ \mathbf{g}$

$$\mathbf{Dom}\left(\mathbf{f} \circ \mathbf{g}\right) = \left\{ \mathbf{x} \in \overbrace{\mathbf{Dom\ g}}^{\mathbb{R}} \;\middle|\; \mathbf{g}(\mathbf{x}) \in \overbrace{\mathbf{Dom\ f}}^{\mathbb{R}} \right\} = \mathbb{R}$$

$$\begin{aligned}
\left(\mathbf{f} \circ \mathbf{g}\right)(\mathbf{x}) \equiv \mathbf{f}(\mathbf{g}(\mathbf{x})) &= 2\left[\mathbf{g}(\mathbf{x})\right] - \left[\mathbf{g}(\mathbf{x})\right]^2 \\
&= 2\left[\mathbf{x}^2 + 2\mathbf{x} + 2\right] - \left[\mathbf{x}^2 + 2\mathbf{x} + 2\right]^2 \\
&= 2\mathbf{x}^2 + 4\mathbf{x} + 4 - \left[\mathbf{x}^4 + 4\mathbf{x}^3 + 8\mathbf{x}^2 + 8\mathbf{x} + 4\right] \\
&= -\mathbf{x}^4 - 4\mathbf{x}^3 - 6\mathbf{x}^2 - 4\mathbf{x}
\end{aligned}$$

Example 06: Consider the functions $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ defined below:

$$\mathbf{f}(\mathbf{x}) = \sqrt{\mathbf{x}-3} \ ; \ \mathbf{g}(\mathbf{x}) = \mathbf{x}^2 - 3\mathbf{x} - 25$$

Find the Composition of $\mathbf{f}(\mathbf{x})$ with $\mathbf{g}(\mathbf{x})$.

Solution:

First, note that

$$\textbf{Dom f} = [3,+\infty) \ ; \ \textbf{Dom g} = \mathbb{R}$$

so that

Composition: $\mathbf{f} \circ \mathbf{g}$

$$\textbf{Dom}(\mathbf{f} \circ \mathbf{g}) = \left\{ \mathbf{x} \in \overbrace{\textbf{Dom g}}^{\mathbb{R}} \ \middle| \ \mathbf{g}(\mathbf{x}) \in \overbrace{\textbf{Dom f}}^{[3,+\infty)} \right\}$$

$$= (-\infty,-4] \cup [7,+\infty)$$

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-4 7

To see this, set $\mathbf{g}(\mathbf{x}) \in [3,+\infty) \Rightarrow \mathbf{g}(\mathbf{x}) \geq 3 \Rightarrow \mathbf{x}^2 - 3\mathbf{x} - 25 \geq 3 \Rightarrow$

$\mathbf{x}^2 - 3\mathbf{x} - 28 \geq 0 \Rightarrow (\mathbf{x}-7)(\mathbf{x}+4) \geq 0 \Rightarrow$

$\mathbf{x} \in (-\infty,-4] \cup [7,+\infty)$

$\therefore \textbf{Dom}(\mathbf{f} \circ \mathbf{g}) = (-\infty,-4] \cup [7,+\infty)$

Also

$$(\mathbf{f} \circ \mathbf{g})(\mathbf{x}) \equiv \mathbf{f}(\mathbf{g}(\mathbf{x})) = \sqrt{[\mathbf{g}(\mathbf{x})] - 3}$$

$$= \sqrt{\left[\mathbf{x}^2 - 3\mathbf{x} - 25\right] - 3}$$

$$= \sqrt{\mathbf{x}^2 - 3\mathbf{x} - 28}$$

