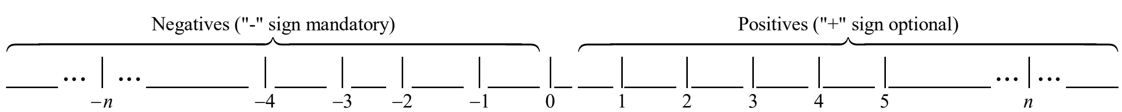
Real Number System

Rational Numbers

(Types, Number Line, Equality, Inequality, Operations, Properties, ...)

Rat = Rational Numbers (aka, Ratio Numbers, Fractions, Q)

It's time to add the "fractions", also called the **ratio**nal numbers, to our current number line:



There are a lot of numbers to add! In general, **rational numbers** have the form $\frac{a}{b}$ where a , b are integers with $b \neq 0$:

$$Rat = \left\{ \frac{a}{b} \overset{\text{Such that}}{|} a, b \text{ are integers with } b \neq 0 \right\}$$

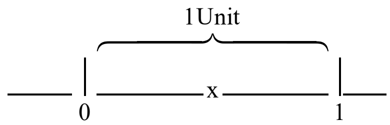
We have already seen that *integers* can be written in this form:

$$4 = \frac{4}{1} = \frac{12}{3} = \frac{24}{6} \ldots$$

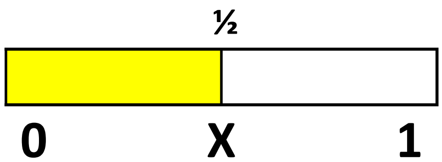
$$3 = \frac{6}{2} = \frac{15}{5} = \ldots$$

$$-7 = \frac{7}{-1} = \frac{-7}{1} = \ldots$$

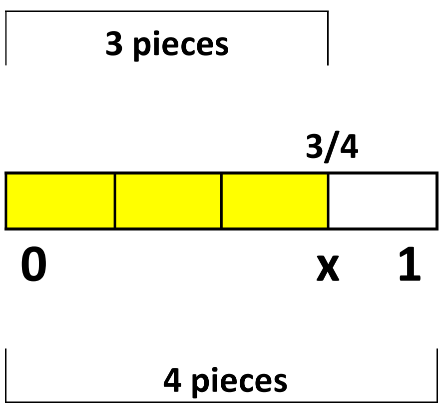
Now, we're ready to define rational numbers (fractions) that are NOT integers. Consider "0" and "1" on the number line:



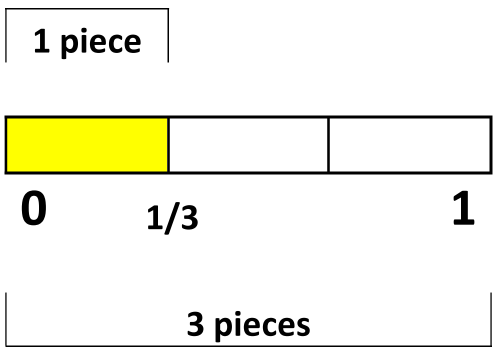
"x" marks the spot for a new number, written $\frac{1}{2}$. It is where the distance between "0" and "1", that is one (1) Unit, is divided into two (2) pieces:



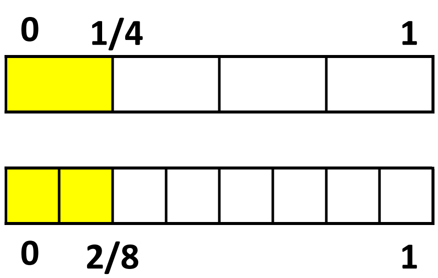
Next consider "0" to "1" being divided into four (4) equal pieces:



Now **x** represents the rational number $\frac{3}{4}$. The number $\frac{1}{3}$ is represented as follows:

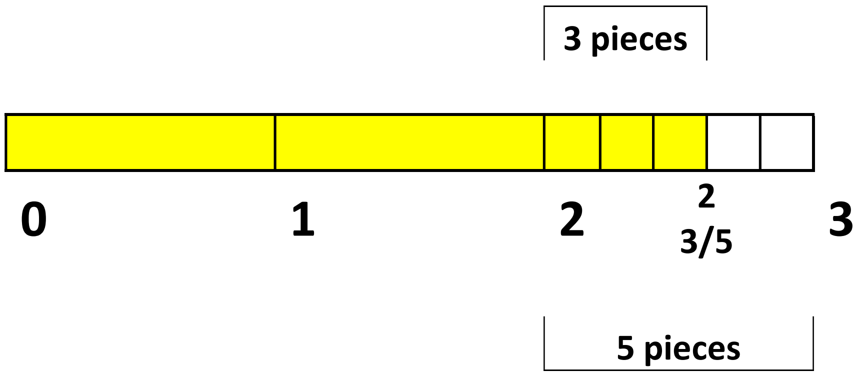


The next figure shows that $\frac{1}{4}$ and $\frac{2}{8}$ represent the same number in a *different* form; they are called **equivalent fractions** and are said to be equal: $\frac{1}{4}=\frac{2}{8}$.

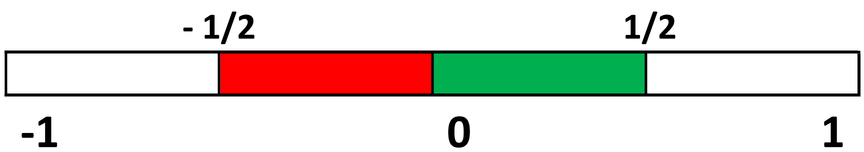


In a similar manner, we can show that $\frac{1}{4}=\frac{2}{8}=\frac{4}{16}=\frac{8}{32}=\ldots$. This means that there are an infinite number of ways to represent a fraction. *Equivalent* is just another way to say that the two (2) fractions are really the *same number* and are located at the same place on the number line.

We can easily represent fractions not between “0” and “1”. For example, if we want to represent $2+\frac{3}{5}$ (usually written $2\frac{3}{5}$ or 2 3/5), we could just construct the following figure:



Fractions can be negative too:

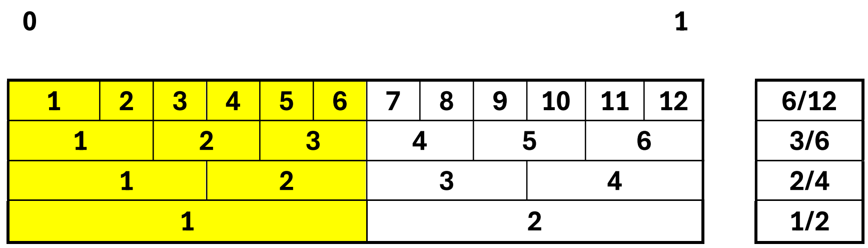


The distances are the same BUT opposite in direction, hence the "-" sign: $-\frac{1}{2}$.

The following diagram represents the fraction $-1\frac{2}{3}$:

Note: $-1-\frac{2}{3}=-(1+\frac{2}{3})=-(1+2/3)=\begin{cases}-1\frac{2}{3}\\ -1\ \ 2/3\end{cases}$

Let's do a little more work with equivalent fractions. Consider the following diagram:



Notice that $\frac{6}{12}=\frac{3}{6}=\frac{2}{4}=\frac{1}{2}$ and then notice that

1. $\frac{6}{12}=\frac{3}{6}$ & $6*6=12*3$
2. $\frac{3}{6}=\frac{2}{4}$ & $3*4=6*2$
3. $\frac{2}{4}=\frac{1}{2}$ & $2*2=4*1$

This example gives rise to the following truth:

$$\frac{a}{b}=\frac{c}{d}\ ;\ b,d\neq 0$$

is equivalent to

$$a*d=b*c$$

Note: Going from $\frac{a}{b}=\frac{c}{d}$ to $a*d=b*c$ is called **cross multiplication**.

Example: Which of the following pairs of fractions are equivalent?

a. $\frac{3}{7}=\frac{4}{5}$; $3*5=15\neq 28=7*4$; Not equivalent
b. $\frac{4}{9}=\frac{12}{27}$; $4*27=108=9*12$; Equivalent
c. $\frac{1}{5}=\frac{2}{7}$; $1*7=7\neq 10=2*5$; Not equivalent

d. $\frac{4}{7}=\frac{12}{21}$; $4*21=84=7*12$; Equivalent

Given a fraction, say $\frac{3}{8}$, we can easily find as many equivalent fractions $\frac{c}{d}$ as we want. If we form $\frac{c}{d}=\frac{3*n}{8*n}=\frac{3}{8}$ for $n=1,2,3,4,\ldots$, we'll get equivalent fractions since 3*(8*n) = 8*(3*n):

1. $\frac{3}{8}$; n = 1
2. $\frac{6}{16}$; n = 2
3. $\frac{9}{24}$; n = 3
4. $\frac{12}{32}$; n = 4
5. …

We now know that two (2) fractions $\frac{a}{b}=\frac{c}{d}$ are equivalent (also said to be equal) when and only when a*d = b*c. If they are NOT equal $(\neq)$, can we determine if $\frac{a}{b}<\frac{c}{d}$ or $\frac{a}{b}>\frac{c}{d}$? The answer is YES! Given $\frac{a}{b},\frac{c}{d}\,(b, d \neq 0)$, we can *always* choose b & d to be positive (b >0 & d >0):

1. $\frac{3}{5}=\frac{a}{b}\Rightarrow b=5>0$
2. $\frac{-4}{-7}=\frac{4}{7}=\frac{a}{b}\Rightarrow b=7>0$
3. $\frac{-2}{11}=\frac{a}{b}\Rightarrow b=11>0$
4. $\frac{5}{-13}=-\frac{5}{13}=\frac{-5}{13}=\frac{a}{b}\Rightarrow b=13>0$

With this in mind,

1. if a d < b c then $\frac{a}{b}<\frac{c}{d}$

2. if a d = b c then $\frac{a}{b} = \frac{c}{d}$

3. if a d > b c then $\frac{a}{b} > \frac{c}{d}$

Example:

1. $\frac{3}{8} \overset{?}{\begin{Bmatrix} < \\ > \end{Bmatrix}} \frac{5}{16}$

   $a\,d = 3*16 = 48 > 40 = 8*5 = b\,c \Rightarrow \frac{3}{8} > \frac{5}{16}$

2. $\frac{2}{7} \overset{?}{\begin{Bmatrix} < \\ > \end{Bmatrix}} \frac{3}{5}$

   $a\,d = 2*5 = 10 < 21 = 7*3 = b\,c \Rightarrow \frac{2}{7} < \frac{3}{5}$

A few more examples follow:

a	b	a/b	c	d	c/d	a*d ? b*c	a/b ? c/d
-1	2	-1/2	-2	3	-2/3	>	>
3	6	3/6	1	2	1/2	=	=
-3	7	-3/7	2	5	2/5	<	<
4	9	4/9	5	7	5/7	<	<
-8	11	-8/11	-9	13	-9/13	<	<

Operations on numbers: New numbers from given numbers

Addition/Subtraction of Rational Numbers – Fractions: Given two (2) fractions $\frac{a}{b}$ and $\frac{c}{b}$ with the *same* denominators, to get their **sum (difference)**, we just add (subtract) their numerators:

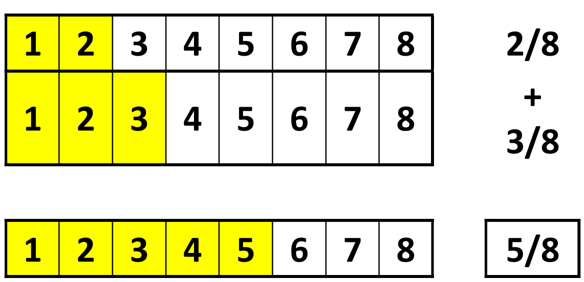
$$\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b} \Rightarrow \begin{cases} \frac{a-c}{b} \\ \frac{a+c}{b} \end{cases}$$

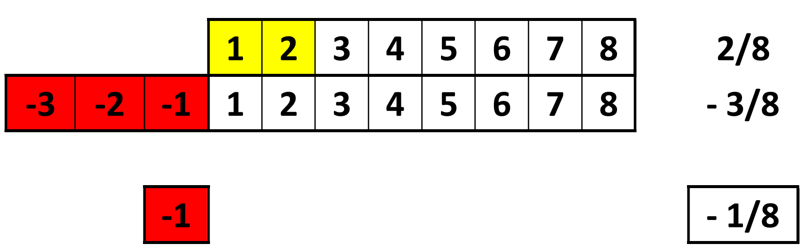
Note: "b" is called the **Common Denominator**. For example,

$$\frac{2}{8} + \frac{3}{8} = \frac{2+3}{8} = \frac{5}{8}$$

and

$$\frac{2}{8} - \frac{3}{8} = \frac{2-3}{8} = \frac{-1}{8} \left(\text{also written as } -\frac{1}{8} \right)$$





Given two (2) fractions with *different* denominators, we have to make their denominators the same using equivalent fractions before we can add or subtract. We can use the following formula, but the *key* is to make their denominators the same:

$$\frac{a}{b} \pm \frac{c}{d} = \frac{a*d}{b*d} \pm \frac{b*c}{b*d} = \frac{a*d \pm b*c}{b*d} \Rightarrow \begin{cases} \dfrac{a*d - b*c}{b*d} \\ \dfrac{a*d + b*c}{b*d} \end{cases}$$

Using this formula, we obtain

$$\frac{2}{5} + \frac{3}{4} = \frac{2*4}{5*4} + \frac{3*5}{4*5} = \frac{8+15}{20} = \frac{23}{20}$$

Notice, we traded $\frac{2}{5}$ and $\frac{3}{4}$ for equivalent fractions that have the *same* denominator:

$$\frac{2}{5} = \frac{8}{20}$$
$$\frac{3}{4} = \frac{15}{20}$$

If we look at the multiples of "4" and "5", we see that "20" is their **Least Common Multiple**: LCM (4,5) = 20

LCM (4,5) = 20

	a 4	b 5
1	4	5
2	8	10
3	12	15
4	16	20
5	20	25
6	24	30
7	28	35
8	32	40
9	36	45

Also, $\frac{2}{5}-\frac{3}{4}=\frac{2*4}{5*4}-\frac{3*5}{4*5}=\frac{8-15}{20}=-\frac{7}{20}$

The product b*d will *always* be a common denominator but using the LCM (b, d) will give us the **Least Common Denominator**. Hence, we'll have smaller numbers to work with! For example, to find $\frac{11}{12}+\frac{17}{18}$, we could use 12*18 = 216 as a *common denominator* but, if we find the LCM (12, 18) = 36 , we'll get the *least common denominator*:

LCM (12, 18) = 36

	a 12	b 18
1	12	18
2	24	36
3	36	54
4	48	72
5	60	90
6	72	108
7	84	126
8	96	144
9	108	162

Recall that we can also find the least common multiple using prime factors:

12 = 2 * 2 * 3

18 = 2 * 3 * 3

so that LCD (12, 18) = 2*2*3*3 = 36

Using the LCD, we obtain

$$\frac{11}{12}+\frac{17}{18}=\frac{11*3}{12*3}+\frac{17*2}{18*2}=\frac{33}{36}+\frac{34}{36}=\frac{67}{36}$$

Note that using the LCD gives us a smart way to find equivalent fractions for $\frac{11}{12}\text{and}\frac{17}{18}$. Let's find $\frac{3}{14}-\frac{1}{10}$. Using prime factors to find the LCD (14,10), we have

14 = 2 * 7

10 = 2 * 5

so that LCD (14, 10) = 70. Now,

$$\frac{3}{14}-\frac{1}{10}=\frac{3*5}{14*5}-\frac{1*7}{10*7}=\frac{15-7}{70}$$
$$=\frac{8}{70}=\frac{2*4}{2*35}=\frac{4}{35}\left(\text{in simplified form}\right)$$

Mixed fractions:

The number, for example, 3 2/5 is called a **mixed fraction.** We can write it in its **fractional form** using the least common denomination process:

$$3\,\frac{2}{5}=3+\frac{2}{5}=\frac{3}{1}+\frac{2}{5}=\frac{3*5}{1*5}+\frac{2}{5}=\frac{17}{5}$$

Also

$$-2\,\frac{3}{7}=-\left(2+\frac{3}{7}\right)=-\left(\frac{2}{1}+\frac{3}{7}\right)=-\left(\frac{2*7}{1*7}+\frac{3}{7}\right)$$
$$=-\frac{17}{7}$$

If we look at this process, we see a quicker way to obtain the fractional form:

$$a\,\frac{c}{d}=\frac{a*d+c}{d}$$

By the way:

1. If the absolute value of the fraction is less than one, it's called a **proper fraction**.

2. If the absolute value of the fraction is greater than one, it's called an **improper fraction**.

Multiplication of Rational Numbers – Fractions: Given two (2) fractions $\frac{a}{b}$ and $\frac{c}{d}$; $b,d \neq 0$, the **product** denoted $\frac{a}{b} * \frac{c}{d}$, is defined by

$$\frac{a}{b} * \frac{c}{d} \overset{Defined}{\equiv} \frac{a*c}{b*d}.$$

Note that we just multiple the numerators and then multiple the denominators. Addition, for example, does NOT work that way:

$$\frac{a}{b} + \frac{c}{d} \neq \frac{a+c}{b+d}$$

Some examples of multiplication follow:

1. $\frac{3}{5} * \frac{2}{7} = \frac{3*2}{5*7} = \frac{6}{35}$
2. $\frac{3}{11} * \frac{2}{3} = \frac{3*2}{11*3} = \frac{6}{33} = \frac{2}{11}$ (we always require simplified form)
3. $\frac{2}{5} * 3 = \frac{2}{5} * \frac{3}{1} = \frac{6}{5} = 1\ 1/5$
4. $2\frac{1}{3} * \frac{4}{5} = \frac{7}{3} * \frac{4}{5} = \frac{28}{15} = \frac{15}{15} + \frac{13}{15} = 1 + \frac{13}{15} = 1\frac{13}{15}$
5. $\frac{2}{3} * \frac{3}{5} + \frac{1}{4} = \frac{2*3}{3*5} + \frac{1}{4} = \frac{6}{15} + \frac{1}{4} = \frac{2}{5} + \frac{1}{4} = \frac{2*4}{5*4} + \frac{1*5}{4*5} = \frac{8}{20} + \frac{5}{20} = \frac{13}{20}$

Division of Rational Numbers – Fractions: Given two (2) fractions $\frac{a}{b}$ and $\frac{c}{d}$; $b,c,d \neq 0$, the **quotient**, denoted $\frac{a}{b} / \frac{c}{d}$ or $\frac{a}{b} \div \frac{c}{d}$ or $\frac{\frac{a}{b}}{\frac{c}{d}}$, is defined by

$$\frac{a}{b} / \frac{c}{d} = \frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} \overset{Defined}{\equiv} \frac{a}{b} * \frac{d}{c}.$$

Before we give examples, we introduce the **reciprocal of a number** $a \neq 0$. It's simply $\frac{1}{a}$:

1. $5 = \frac{5}{1}$: Reciprocal $= \frac{1}{5}$
2. $\frac{3}{7}$: Reciprocal $= \frac{1}{\frac{3}{7}} = \frac{\frac{1}{1}}{\frac{3}{7}} = \frac{1}{1} * \frac{7}{3} = \frac{7}{3}$ (**Note:** we just "flipped" the fraction $\frac{3}{7} \rightarrow \rightarrow \frac{7}{3}$)
3. $-\frac{2}{5}$: Reciprocal $= \frac{1}{-\frac{2}{5}} = \frac{\frac{1}{1}}{-\frac{2}{5}} = \frac{1}{1} * \left(-\frac{5}{2}\right) = -\frac{5}{2}$
4. $\frac{c}{d}$: Reciprocal $= \frac{1}{\frac{c}{d}} = \frac{\frac{1}{1}}{\frac{c}{d}} = \frac{1}{1} * \frac{d}{c} = \frac{d}{c}$; $d \neq 0$

Note that $a * \frac{1}{a} = \frac{a}{1} * \frac{1}{a} = \frac{a}{a} = 1$; $a \neq 0$

With the reciprocal in mind the quotient of $\frac{a}{b}$ and $\frac{c}{d}$; $b, c, d \neq 0$ is the numerator fraction multiplied by the reciprocal of the denominator fraction:

$$\frac{a}{b} \,/\, \frac{c}{d} = \quad \frac{a}{b} \;\div\; \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} * \frac{d}{c}$$

Now for some examples:

1. $\frac{4}{3}$ divided by $3 = \frac{3}{1}$ equals $\frac{4}{3} * \frac{1}{3} = \frac{4}{9}$
2. $-\frac{3}{7} \div \frac{2}{5} = -\frac{3}{7} * \frac{5}{2} = -\frac{3*5}{7*2} = -\frac{15}{14} = -1\,\frac{1}{14}$
3. $-\frac{1}{6} / \frac{5}{2} = -\frac{1}{6} * \frac{2}{5} = -\frac{1*2}{6*5} = -\frac{2}{30} = -\frac{1}{15}$ (in simplified form)

4. $-\dfrac{-\frac{3}{7}}{\frac{13}{14}} = \left(-\dfrac{3}{7}\right)*\left(-\dfrac{14}{13}\right) = \dfrac{3*14}{7*13} = \dfrac{3*2*7}{7*13} = \dfrac{6}{13}$

Exponentiation:

NEVER allowing division by zero and always excluding 0^0, we can update our **power number** definitions to include rational numbers:

$$\text{Base}^{\text{Exponent}} = \left(\frac{a}{b}\right)^n \overset{defined}{=} \overbrace{\left(\frac{a}{b}\right)*\left(\frac{a}{b}\right)*\left(\frac{a}{b}\right)*\ldots*\left(\frac{a}{b}\right)}^{\text{n times}} = \frac{a^n}{b^n}$$

where “a” and “b” are integers and “n” is a positive integer or zero. Now we can also allow the “Exponent” to be negative; assuming that “n” is a positive integer, we define

$$\text{Base}^{\text{Exponent}} = \left(\frac{a}{b}\right)^{-n} \overset{defined}{=} \left(\frac{b}{a}\right)^n$$

If b =1, we have $a^{-n} = \dfrac{1}{a^n}\ ;\ \dfrac{1}{a^{-n}} = a^n$

Several examples follow:

1. $\left(\dfrac{3}{4}\right)^2 = \left(\dfrac{3}{4}\right)*\left(\dfrac{3}{4}\right) = \dfrac{9}{16} = \dfrac{3^2}{4^2}$

2. $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

3. $\left(\frac{5}{2}\right)^{-1} = \left(\frac{2}{5}\right)^{1} = \frac{2}{5}$

4. $\frac{1}{3^{-3}} = 3^{3} = 27$

5. $\left(\frac{2}{3}\right)^{-4} = \left(\frac{3}{2}\right)^{4} = \frac{3^4}{2^4} = \frac{81}{16}$

6. $\left(\frac{4}{5}\right)^{3} = \frac{4^3}{5^3} = \frac{64}{125}$

7. $\left(-\frac{3}{7}\right)^{2} = \left(-\frac{3}{7}\right) * \left(-\frac{3}{7}\right) = \frac{9}{49}$

or

$$\left(-\frac{3}{7}\right)^{2} = \left(\frac{-3}{7}\right)^{2} = \frac{(-3)^2}{7^2} = \frac{9}{49}$$

Note

$\left(\frac{a}{b}\right)^{n} \overset{defined}{=} \frac{a^n}{b^n}$; a,b are integers with $b \neq 0$; n is a positive integer

$\left(\frac{a}{b}\right)^{-n} \overset{defined}{=} \left(\frac{b}{a}\right)^{n}$; $a, b \neq 0$

Decimal Representation of Fractions:

We have seen that the integers can be written in **fractional form:**

$$5 = \frac{5}{1}\ ;\ 17 = \frac{17}{1}\ ;\ -11 = -\frac{11}{1}\ ...$$

They are easily written in **decimal form**:

$$5. = \frac{5.}{1}\ ;\ 17. = \frac{17.}{1}\ ;\ -11. = -\frac{11.}{1}\ ...$$

where the "." at the right-hand side and bottom of the integer is called the **decimal point**.

Note that $4 \cdot 8 = 32$ but 4.8 does NOT mean multiplication.

Non-integer fractions also have a **decimal representation** (**decimal form**) which we now discuss. First consider an integer such as 2734 (= 2734.). We can write it as

2734 = 2*1000 + 7*100 + 3*10 + 4

In *chart form*, we have

Sign	Thousands (*1000)	Hundreds (*100)	Tens (*10)	Units	Decimal Point
+	**2**	**7**	**3**	**4**	**.**

Here are a few more examples using integers:

Sign	10 Thousands (*10000)	Thousands (*1000)	Hundreds (*100)	Tens (*10)	Units	Decimal Point
+		2	7	3	4	.
-	3	0	1	2	5	.
+				7	3	.
-		4	8	2	4	.
+	6	8	7	5	0	.

We usually use a comma “,” to separate “thousands”:

243765887. = 243,765,887.

When we deal with fractions, we frequently need tenths, hundredths, thousandths, ... : $\frac{1}{10}$; $\frac{1}{100}$; $\frac{1}{1000}$; $\frac{1}{10000}$; ... : Here is a chart for the decimal number 53724.569:

Sign	10 Thousands (*10000)	Thousands (*1000)	Hundreds (*100)	Tens (*10)	Units	Decimal Point	Tenths (*1/10)	Hundredths (*1/100)	Thousands (*1/1000)
+	5	3	7	2	4	.	5	6	9

Thus 53724.569 = 5*10000 + 3*1000 + 7*100 + 2*10 + 4 + 5*1/10

+ 6*1/100 + 9*1/1000

$= \frac{53724569.}{1000}$ (Common denominator = 1000)

Here are some decimal numbers with their fractional equivalents:

1. $2.34 = \frac{2}{1} + \frac{3}{10} + \frac{4}{100} = \frac{2*100}{1*100} + \frac{3*10}{10*10} + \frac{4}{100} = \frac{200+30+4}{100} = \frac{234.}{100}$
2. $-0.2 = -\frac{2.}{10}$
3. $0.45 = \frac{4}{10} + \frac{5}{100} = \frac{4*10}{10*10} + \frac{5}{100} = \frac{45.}{100}$

Multiplication and Division by Powers of 10:

Multiplication:

Notice that

1. $5.000 * 10^1 = 50.000$
2. $5.000 * 10^2 = 5.000 * 100 = 500.000$
3. $5.000 * 10^3 = 5.000 * 1000 = 5000.000$
4. …

What the examples are telling us to do is to move the decimal point to the "***right***" the same number of places as the power (1,2,3,...).

Division:

Notice that

1. $\frac{5.000}{10^1} = 0.5000 \left(\text{since } 0.5000 * 10 = 5.000\right)$
2. $\frac{5.000}{10^2} = \frac{5.000}{100} = 0.05000 \left(\text{since } 0.05000 * 100 = 5.000\right)$
3. $\frac{5.000}{10^3} = \frac{5.000}{1000} = 0.005000 \left(\text{since } 0.005000 * 1000 = 5.000\right)$
4. …

What the examples are telling us to do is to move the decimal point to the "***left"*** the same number of places as the power (1,2,3, ...).

Caution: We must be careful where the decimal point is and what operations we are asked to perform:

$$4 * 3 + 4^3 + 4.3 = 12 + 64 + 4.3 = 80.3$$

Decimal form *to* Fractional form and Fractional form *to* Decimal form:

We have seen how to go from decimal form to fraction form:

$$3.4 = 3 + \frac{4}{10} = \frac{3}{1} + \frac{4}{10} = \frac{3*10}{1*10} + \frac{4}{10} = \frac{34}{10} = \frac{17*2}{5*2} = \frac{17}{5}$$

Going from fractional form to decimal form involves division. We have seen division before:

$$\frac{12}{3} = 4$$

since 12 = 3*4. Another way to obtain this is to use **long division** which has the form:

$$\text{Divisor}\overline{\smash{)}\text{Dividend}}^{\ \text{Quotient}} \qquad ; \qquad \frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}$$

$$\begin{array}{l} . \\ . \\ \underline{\cdots} \\ \quad \text{Remainder} \end{array}$$

We have

$$\begin{array}{r l} & \quad 4 \\ \text{Times } 3\overline{\smash{)}12} & \\ \underline{12} & \text{Subtract ; note } 3*4 = 12 \\ 0 & \text{Remainder} \end{array}$$

Thus, $\frac{12}{3} = 4$.

However, with $\frac{17}{5}$ there is NOT an integer “a” such that 17 = 5*a:

$5*1=5$ to small

$5*2=10$ to small

$5*3=15$ to small

$5*4=20$ to big

The number a = 3 is “close”:

17. = 17 = 5*3 + 2

We interpret this as follows:

```
          3.
Times 5)17.0
        15 Subtract; note 3*5 =15
         2
```

```
         3.4
Times 5)17.0
        15
         20  Bring down a "0"
         20  Subtract ; note 4*5 =20
          0 Remainder
```

Thus, $\frac{17}{5}=3.4$ since 17. = 5*3.4. To check this, note that

$$3.4=3+0.4=\frac{3}{1}+\frac{4}{10}=\frac{3*10}{1*10}+\frac{4}{10}=\frac{30}{10}+\frac{4}{10}=\frac{34}{10}=\frac{17}{5}$$

Important: Note how the decimal points above were “aligned”.

Let’s find the decimal form of 29.61 divided by 7, that is $\frac{29.61}{7}$. We have

$$\begin{array}{r} 4. \\ \text{Times } 7\overline{)29.61} \end{array}$$

$\underline{28}$ Subtract ; note 4*7 = 28

1

$$\begin{array}{r} 4.2 \\ \text{Times } 7\overline{)29.61} \end{array}$$

$\underline{28}$

16 Bring down the "6"

$\underline{14}$ Subtract ; note 3*7 = 14

2

$$\begin{array}{r} 4.23 \\ \text{Times } 7\overline{)29.61} \end{array}$$

$\underline{28}$

16

$\underline{14}$

21 Bring down the "1"

$\underline{21}$ Subtract ; note 3*7 = 21

0 Remainder

Thus, $\frac{29.61}{7} = 4.23$. Let's check this one:

$$7 * 4.23 = 7\left(4 + \frac{2}{10} + \frac{3}{100}\right) = 28 + \frac{14}{10} + \frac{21}{100} = 28 + \left(\frac{10+4}{10}\right) + \left(\frac{20+1}{100}\right)$$

$$= 28 + \left(1 + \frac{4}{10}\right) + \left(\frac{2}{10} + \frac{1}{100}\right) = 29.61$$

Here are a few more examples:

1. $\frac{306}{5} = ?$

$$\begin{array}{l} \quad\quad\quad\;\; 6 \\ \text{Times } 5\overline{)306.0} \\ \quad\quad\quad\; \underline{30} \;\; \text{Subtract ; note } 6*5 = 30 \\ \quad\quad\quad\quad\; 6 \;\; \text{Bring down the "6"} \end{array}$$

$$\begin{array}{l} \quad\quad\quad\; 61 \\ \text{Times } 5\overline{)306.0} \\ \quad\quad\quad\; \underline{30} \\ \quad\quad\quad\quad\; 6 \\ \quad\quad\quad\quad\; \underline{5} \;\; \text{Subtract ' } 1*5 = 5 \\ \quad\quad\quad\quad\; 10 \;\; \text{Bring down the "0"} \end{array}$$

$$\begin{array}{l} \quad\quad\quad\;\; 61.2 \\ \text{Times } 5\overline{)306.0} \\ \quad\quad\quad\; \underline{30} \\ \quad\quad\quad\quad\; 6 \\ \quad\quad\quad\quad\; \underline{5} \\ \quad\quad\quad\quad\; 10 \\ \quad\quad\quad\quad\; \underline{10} \;\; \text{Subtract ; note } 2*5 = 10 \\ \quad\quad\quad\quad\;\; 0 \;\; \text{Remainder} \end{array}$$

2. $\frac{825}{3} = ?$ We will NOT put all of the reminders in from now on.

$$\begin{array}{l} \quad 275.2 \\ 3\overline{)825.6} \\ \;\; \underline{6} \\ \;\; 22 \\ \;\; \underline{21} \\ \quad\;\; 15 \\ \quad\;\; \underline{15} \\ \quad\quad\quad 6 \\ \quad\quad\quad \underline{6} \\ \quad\quad\quad 0 \end{array}$$

Notice that every quotient so far terminated, that is, it ended in zeroes:

- 4.00…
- 3.400…
- 4.2300…
- 61.200…
- 275.200…

Fractions may also have a repeating patten other than zeroes which we now explore. Consider the fraction $\frac{2}{3}$. Using long division, we obtain

$$\text{Times } 3\overline{)\,2.0000}\quad \text{quotient } 0.666...$$

$$\begin{array}{l} \underline{18} \\ \;\;20 \\ \;\;\underline{18} \\ \;\;\;\;20 \\ \;\;\;\;\underline{18} \\ \;\;\;\;\;\;20 \\ \;\;\;\;\;\;\underline{18} \\ \;\;\;\;\;\;\;\;20 \\ \;\;\;\;\;\;\;\;... \end{array}$$

Notice that the pattern

$$\begin{array}{l} 20 \\ \underline{18} \\ \;\;20 \end{array}$$

repeats forever so we write $\frac{2}{3} = 0.66666... \overset{\text{OR}}{=} 0.6666\overline{6}$. The three dots

(...) or the bar ($\overline{}$) are the notations meaning that the pattern continues for ever. Getting the fraction from the decimal form is a little more involved. Since the decimal repeats $0.6666\overline{6}$ after only one digit (1), we do the following calculation, letting “d” represent the decimal: $d = 0.6666\overline{6}$

$10d = 6.6666\overline{6}$ Recall that multiplication by 10 moves the decimal point one place to the right

$$\frac{d = 0.6666\overline{6}}{}$$ Subtract

$9d = 6$

$$d = \frac{6}{9} = \frac{2}{3}$$

What fraction does $d = 2.141414\overline{14}$ represent? Since the decimal repeats after two (2) digits, we multiple by 100:

$100d = 214.1414\overline{14}$ Recall that multiplication by 100 moves the decimal point two places to the right

$d = 2.1414\overline{14}$ Subtract

$99d = 212$

$$d = \frac{212}{99}$$