# Real Number System

## Irrational Numbers

## IrRat = Irrational Numbers (Non-rational Numbers, Non-fractions)

After the rational numbers (fractions) were constructed and placed on the number line,  the "Math Gurus" thought that every point on the number line could be represented by a fraction. They were wrong! A brilliant man named Pythagoras came along and proved that there were an infinite number of "other" numbers that were NOT fractions. These other numbers were called **irrational numbers**. In "real life" applications, these irrational numbers play an important role.

We have seen that **rational numbers**, that is fractions, have decimal expansions that

1. terminate: $\frac{1}{2} = 0.5 = 0.500000000...$

or

2. repeat: $\frac{2}{3} = 0.6666\overline{6} = 0.66666...$

Note: These three (3) dots **DO** imply a repeating pattern.

The **irrational numbers** also have decimal expansions, but they do NOT terminate nor repeat:

For example,

$\pi = 3.141592653589879...$

$e = 2.71828182845904...$

5.101001000100001000001...

$\sqrt{2} = 1.41421356237309...$ (called the **square root of 2** ; discussed below)

Note: These three (3) dots **DO NOT** imply a repeating pattern.

When we operate (+,-,*,/, ...) with rational numbers (fractions), we get *exact* answers ; when we operate with irrational numbers, we only get *approximate* answers.

We have previously discussed **power numbers** of the for

$$\left(\frac{a}{b}\right)^n$$

where "$\frac{a}{b}$" is a rational number and "n" is an integer ( $0^0$ is NOT allowed).

Now, we consider **power numbers** of the form

$$\left(\frac{a}{b}\right)^r = a^r$$

where

1. b = 1
2. "r" is the fraction 1/2 or 1/3

Let us consider r = 1/2 first: $a^{1/2}$ with $a \geq 0$. Note that $a^{1/2}$ is frequently written as $a^{1/2} = \sqrt[2]{a} = \sqrt{a}$ and is called the **square root of** "a". The $\sqrt{a}$ represents a non-negative number "c" such that $c^2 = a$. There are some *nice* square roots:

1. $\sqrt{0} = 0$ since 0*0 = 0
2. $\sqrt{1} = 1$ since 1*1 = 1
3. $\sqrt{4} = 2$ since 2*2 = 4
4. $\sqrt{9} = 3$ since 3*3 = 9
5. $\sqrt{16} = 4$ since 4*4 = 16
6. $\sqrt{25} = 5$ since 5*5 = 25
7. $\sqrt{36} = 6$ since 6*6 = 36
8. …

Notice that for every positive integer “a”

$$\sqrt{a^2} = a$$

The reason we have postponed talking about square roots until now is that they frequently result in irrational numbers, that is, numbers whose decimal expansions (representations, forms) do not terminate nor repeat. In the past, tables were constructed that contained decimal approximations of square roots that resulted in irrational numbers. The good news is that now we can use a “scientific calculator” to get approximate values easily – we just use the “$\sqrt{\ }$” key. For example,

$$\sqrt{2} \approx 1.414214$$

to six (6) decimals. A few more square root approximations are given below:

$$\sqrt{3} \approx 1.732051$$

$$\sqrt{5} \approx 2.236068$$

$$\sqrt{17} \approx 4.123106$$

$$\sqrt{23} \approx 4.795832$$

Now consider r = 1/3: $a^{1/3}$ where “a” is now any integer. Note that $a^{1/3}$ is frequently written as $a^{1/3} = \sqrt[3]{a}$ and is called the **cube root of** “a”. The $\sqrt[3]{a}$ represents a number “c” such that $c^3 = a$. Here are some *nice* cube roots:

1. $\sqrt[3]{8} = 2$ since $2^3 = 8$
2. $\sqrt[3]{-8} = -2$ since $(-2)^3 = -8$
3. $\sqrt[3]{-125} = -5$ since $(-5)^3 = -125$

Note that whereas $\sqrt{a} \geq 0$, $\sqrt[3]{a}$ can be negative, zero, or positive.

Not so nice ones (6 decimals) resulting in irrational numbers include

1. $\sqrt[3]{11} \approx 2.223980$
2. $\sqrt[3]{-19} \approx -2.668402$
3. $\sqrt[3]{41} \approx 3.448217$

We can also find square and cube roots of "$\frac{a}{b}$": $\sqrt{\frac{a}{b}}, \left(\frac{a}{b} \geq 0\right); \sqrt[3]{\frac{a}{b}}$

1. $\sqrt{\frac{16}{9}} = \frac{4}{3}$ since $\left(\frac{4}{3}\right)^2 = \frac{16}{9}$ ; "nice"
2. $\sqrt{\frac{29}{5}} \approx 2.408318$ ; "NOT nice"
3. $\sqrt[3]{\frac{64}{27}} = \frac{4}{3}$ since $\left(\frac{4}{3}\right)^3 = \frac{64}{27}$
4. $\sqrt[3]{\frac{1}{8}} = \frac{1}{2}$ since $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$
5. $\sqrt[3]{\frac{5}{7}} \approx 0.893904$
6. $\sqrt[3]{\frac{23}{13}} \approx 1.209469$

Actually, $\text{Base}^{\text{Exponent}}$ can frequently be defined when both the Base and Exponent are rational or irrational numbers. For now, we will just use the "^" key on a scientific calculator to obtain these values, which are frequently only approximate.