Equations – Linear

"$x^1 = x$" is the unknown

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Recall the definition of **Algebra**:

> **Algebra** is arithmetic with letters (aka, applied arithmetic)

and *every* letter represents a number.
Recall the definition of **Algebraic Expression:**

> An **algebraic expression** is a numerical expression with letters that represent numbers

Now we consider two (2) expressions separated by an equal ("=") sign:
An **equation** consists of an equal ("=") sign with an expression on its left-hand side (LHS) and an expression on its right-hand side (RHS):

> **LHS = RHS**

Initially, there is a letter, say "x", on at least one side of the equation which is called the **unknown**. The goal is to find all the values, if any, for "x" that make the equation true. This means that when the value for "x" is substituted into the equation, the value of the expression of the LHS *equals* the value of the expression on the RHS. This value of "x" is called a **solution** of the equation. Consider, for example, for the equation $2x-4=11-x$. The number x = 5 is a solution since

$$2[5]-4=6=11-[5]$$

However, x = 3 is NOT a solution since

$$2[3]-4=2\neq 8=11-[3]$$

The number of solutions an equation possesses depends on its type. We now consider **linear equations** which have "x" only raised to the first power: $x = x^1$. Learning how to solve various equations (aka, equalities ("=")), including linear equations, will require the use of equality properties we now list:

Equality Properties:

Let “a, b & c” be numbers. If a = b, then

1. a + c = b + c
2. a – c = b – c
3. a*c = b*c
4. a/c = b/c ; c ≠ 0

With only one letter, say “x”

To solve a **linear equation**, we use properties 1 – 4 on both sides until “x” is isolated: x = #. Let’s look at some examples:

1. Solve x = 2

   This linear equation is already solved; “x” is isolated: x = 2. We graph this solution on the horizontal number line:

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   Solution (s) 2

2. Solve x – 5 = 3

   Adding “5” to both sides, we obtain

   $$x-5=3$$
   $$x\overbrace{-5+5}^{0}=3+5$$
   $$x+0=8$$
   $$x=8$$

   Geometrically, we have

   ________________ 0 _________________ |_______

   Solution (s) 8

3. Solve 4x + 3 = 11

We have

$$4x + 3 = 11$$
$$4x \overbrace{+3 - 3}^{0} = 11 - 3$$
$$4x + 0 = 8$$
$$4x = 8$$
$$\frac{4x}{4} = \frac{8}{4}$$
$$x = 2$$

Geometrically, we have

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Solution (s) 2

4. Solve 5 – x = 3x + 20

$$5 - x = 3x + 20$$
$$5 \overbrace{-x + x}^{0} = 3x + 20 + x = 3x + 1x + 20$$
$$5 = 4x + 20$$
$$5 - 20 = 4x \overbrace{+20 - 20}^{0}$$
$$-15 = 4x + 0$$
$$-15 = 4x$$
$$-\frac{15}{4} = \frac{\cancel{4}x}{\cancel{4}}$$
$$-\frac{15}{4} = x \left(\text{or } x = -\frac{15}{4}\right)$$

Geometrically, we have

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Solution (s) .... −15/4

5. Solve 3 + (4 – x) = 2(x + 5)

$$3+(4-x)=2(x+5)$$
$$3+4-x=2x+10$$
$$7\overbrace{-x+x}^{0}=2x+10+x=2x+1x+10$$
$$7+0=3x+10$$
$$7=3x+10$$
$$7-10=3x+\overbrace{10-10}^{0}=3x+0$$
$$-3=3x$$
$$-\frac{3}{3}=\frac{\cancel{3}x}{\cancel{3}}$$
$$-1=x\left(\text{or } x=-1\right)$$

Geometrically, we have

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Solution (s) -1

6. Solve 2(4x + 5) = 2(x +1) – 4

$$2(4x+5)=2\overset{\text{Start}}{(}\ x+1\overset{\textit{Stop}}{)}-4$$
$$8x+10=2x+2-4=2x-2$$
$$8x+\overbrace{10-10}^{0}=2x-2-10=2x-12$$
$$8x=2x-12$$
$$8x-2x=2x-12-2x=-12+\overbrace{2x-2x}^{0}=-12$$
$$6x=8x-2x=-12$$
$$\frac{6x}{6}=-\frac{12}{6}$$
$$x=-2$$

Geometrically, we have

$$\underset{\text{Solution (s)}}{\_\_\_\_\_\_\_\_\_\_}\ \underset{-2}{|}\ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\ 0\ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

For practice, we'll *check* our "potential solution":

$$2(\overbrace{4*(-2)}^{-8}+5)\stackrel{?}{=}2\overbrace{(-2+1)}^{-1}-4$$
$$2*(-3)=-2-4$$
$$-6\stackrel{\text{YES}}{=}-6$$

7. Solve 1 + 4x = 13/4 + x

$$1+4x=\frac{13}{4}+x$$

$$1+\overbrace{4x-x}^{3x}=\frac{13}{4}+\overbrace{x-x}^{0}$$

$$3x\overbrace{+1-1}^{0}=\frac{13}{4}-1=\frac{13}{4}-\frac{4}{4}=\frac{9}{4}$$

$$3x=\frac{9}{4}$$

$$\frac{\cancel{3}x}{\cancel{3}}=\frac{\frac{9}{4}}{3}=\frac{\frac{9}{4}}{\frac{3}{1}}=\frac{9}{4}*\frac{1}{3}=\frac{3}{4}$$

$$x=\frac{3}{4}$$

Geometrically, we have

$$\underset{\text{Solution (s)}}{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}\ 0\ \_\_\ \underset{3/4}{|}\ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

For practice, we'll *check* our "potential solution":

$$1+4*\frac{3}{4}\overset{?}{=}\frac{13}{4}+\frac{3}{4}=\frac{16}{4}$$

$$1+3\overset{?}{=}4$$

$$4\overset{\text{YES}}{=}4$$

8. Solve $0.1x-1.3=2(1.4-0.21x)$ … *decimal* coefficients

Going to "trade" for *fractional* coefficients

$$\frac{1}{10}x-\frac{13}{10}=2.8-0.42x=\frac{28}{10}-\frac{42}{100}x$$

Going to "trade" for i*nteger* coefficients

$$100\left(\frac{1}{10}x-\frac{13}{10}\right)=100\left(\frac{28}{10}-\frac{42}{100}x\right)$$; multiple by 100

$$10x-130=280-42x$$

$$10x+42x=280+130$$; did a few steps in my head

$$52x=410$$

$$x=\frac{410}{52}=\frac{205}{26}$$

Geometrically, we have

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Solution (s) 205/26

With more than one letter (aka, Literal Equations) ... we must be told which letter to solve for.

Example 1: The solution $\mathbf{x}$ of the equation $\frac{\mathbf{a}}{\mathbf{x}} = \frac{\mathbf{b}}{\mathbf{a}}$ is x = ?

Solution:

Step	Equation	Reason
0	$\frac{\mathbf{a}}{\mathbf{x}} = \frac{\mathbf{b}}{\mathbf{a}}$	
1	$\mathbf{a}^2 = \mathbf{bx}$	
2	$\frac{\mathbf{a}^2}{\mathbf{b}} = \mathbf{x}$ **OR** $\mathbf{x} = \frac{\mathbf{a}^2}{\mathbf{b}}$	

Note: Can not graph the solution set.

Example 2: The solution $\mathbf{x}$ of the equation $\mathbf{a} - \mathbf{bx} = \mathbf{c}(2 - \mathbf{x})$ is x = ?

Solution:

Step	Equation	Reason
0	$\mathbf{a} - \mathbf{bx} = \mathbf{c}(2 - \mathbf{x})$	
1	$\mathbf{a} - \mathbf{bx} = 2\mathbf{c} - \mathbf{cx}$	
2	$\mathbf{a} - 2\mathbf{c} = \mathbf{bx} - \mathbf{cx}$	Group all the x terms together
3	$\mathbf{a} - 2\mathbf{c} = \mathbf{x}(\mathbf{b} - \mathbf{c})$	Factor
4	$\frac{\mathbf{a} - 2\mathbf{c}}{\mathbf{b} - \mathbf{c}} = \mathbf{x}$ **OR** $\mathbf{x} = \frac{\mathbf{a} - 2\mathbf{c}}{\mathbf{b} - \mathbf{c}} = \frac{2\mathbf{c} - \mathbf{a}}{\mathbf{c} - \mathbf{b}}$	Divide ; isolate "x"

POWER of Algebra: Solve the following equations:

a. $2\mathbf{x} = 3 \Rightarrow \mathbf{x} = \frac{3}{2}$

b. $-4\mathbf{x} = 7 \Rightarrow \mathbf{x} = -\frac{7}{4}$

c. … There are an infinite number of equations like these. However, consider $\mathbf{ax} = \mathbf{b}$; $\mathbf{a} \neq 0$. We have

$$\mathbf{ax} = \mathbf{b} \Rightarrow \mathbf{x} = \frac{\mathbf{b}}{\mathbf{a}}$$

Note: We have actually solved an infinite number of equations. This is the power of algebra!

Additional Linear Equations with "x":

Example 1: Find the solution of the equation $3-(4-2\mathbf{x})=3(\mathbf{x}+2)-4\mathbf{x}+2$

Solution:

Step	Equation	Reason
0	$3-(4-2\mathbf{x})=3(\mathbf{x}+2)-4\mathbf{x}+2$	
1	$3-4+2\mathbf{x}=3\mathbf{x}+6-4\mathbf{x}+2$	
2	$-1+2\mathbf{x}=-\mathbf{x}+8$	
3	$\mathbf{x}+2\mathbf{x}=8+1$	
4	$3\mathbf{x}=9$	
5	$\mathbf{x}=3$	

Solution graph: ________|_0________3____________

Example 2: Solve for $\mathbf{x}$ in the equation $2(5\mathbf{x}-3)=7-2\mathbf{x}$

Solution:

Step	Equation	Reason
0	$2(5\mathbf{x}-3)=7-2\mathbf{x}$	
1	$10\mathbf{x}-6=7-2\mathbf{x}$	
2	$2\mathbf{x}+10\mathbf{x}=6+7$	
3	$12\mathbf{x}=13$	
4	$\mathbf{x}=\frac{13}{12}$	

Solution graph:

________|_0________13/12____________

Example 3: Find the solution $\mathbf{x}$ in the equation $\frac{3}{4}\mathbf{x}-2=\frac{1}{3}+2\mathbf{x}$

Solution:

Step	Equation	Reason
0	$\frac{3}{4}\mathbf{x}-2=\frac{1}{3}+2\mathbf{x}$	
1	$12\left(\frac{3\mathbf{x}}{4}-\frac{2}{1}\right)=12\left(\frac{1}{3}+\frac{2\mathbf{x}}{1}\right)$	Eliminate fractions with common denominator
2	$9\mathbf{x}-24=4+24\mathbf{x}$	
3	$9\mathbf{x}-24\mathbf{x}=24+4$	
4	$-15\mathbf{x}=28$	
5	$\mathbf{x}=-\frac{28}{15}$	

Solution graph: ___________ $-28/15$ ____________ | _____________ 0

Example 4: Solve for $\mathbf{x}$: $\frac{2}{1-\mathbf{x}}=\frac{4}{3}$

Solution:

Step	Equation	Reason
0	$\frac{2}{1-\mathbf{x}}=\frac{4}{3}$	$\mathbf{x}\neq 1$
1	$2*3=4(1-\mathbf{x})$	Cross Multiple
2	$6=4-4\mathbf{x}$	
3	$4\mathbf{x}=4-6$	
4	$4\mathbf{x}=-2$	
5	$\mathbf{x}=-\frac{2}{4}=-\frac{1}{2}$	

Solution graph: ___________ $-1/2$ ____________ | _____________ 0

Note: This equation is a linear "rational" equation since the "x" is in the denominator, but it can be converted to the usual linear format.