Numbers – Complex
Two Part Numbers
Real Part & Imaginary Part

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Complex Numbers are two (2) part numbers:

$$z = Part\ \#1 + i\ Part\ \#2$$
$$= a + i\ b = (a,b)$$

where a & b are real numbers and **"i"** is called the **imaginary unit** – lousy name. Examples of complex numbers that we will use below:

$$z_1 = a + ib = 2 + 3i$$
$$z_2 = c + id = -1 + 5i$$

Imaginary Unit Definition: $i = \sqrt{-1}$

We have the following:

$$i = i^1 = \sqrt{-1}\ \text{Definition}$$
$$i^2 = \left(\sqrt{-1}\right)^2 = \left[(-1)^{1/2}\right]^2 = -1$$
$$i^3 = i^2 * i^1 = -1 * i = -i$$
$$i^4 = i^2 * i^2 = (-1)*(-1) = 1$$
$$i^5 = i^4 * i^1 = i$$
$$i^6 = i^4 * i^2 = -1$$
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Key: Positive Integer "I" can be written as $I = 4Q + R$ where Q is the Quotient and R is the Remainder satisfying $0 \le R \le 3$. Hence,

$$i^I = i^{4Q+R} = (i^4)^Q * i^R = 1^Q * i^R = \begin{cases} 1 \; ; R = 0 \\ i \; ; R = 1 \\ -1 \; ; R = 2 \\ -i \; ; R = 3 \end{cases}$$

The good news is that complex numbers satisfy the exponential and other properties that real numbers satisfy!

Example 1: $\mathbf{i}^{79} = ?$
Solution:

Step	Equation	Reason
0	$\mathbf{i}^{79} =$	
1	$\mathbf{i}^{4*19+3} =$	
2	$\left(\mathbf{i}^4\right)^{19} \mathbf{i}^3 =$	
3	$(1)^{19} \mathbf{i}^3 =$	
4	$\mathbf{i}^3 =$	
5	$-\mathbf{i}$	

Example 2: $\dfrac{1}{i^{19}} = ?$

Solution:

Step	Equation	Reason
0	$\dfrac{1}{i^{19}} =$	
1	$\dfrac{1}{i^{4*4+3}} =$	
2	$\dfrac{1}{\left(i^4\right)^4 i^3} =$	
3	$\dfrac{1}{(1)^4 i^3} =$	$i^4 = 1$
4	$\dfrac{1}{i^3} =$	
5	$\dfrac{1}{-i} =$	
6	$-\dfrac{1}{i} * \dfrac{i}{i} =$	
7	$-\dfrac{i}{i^2} =$	
8	$-\dfrac{i}{-1} =$	
9	i	

Example 3: $\mathbf{i}^{-173} = ?$
Solution:

Step	Equation	Reason
0	$\mathbf{i}^{-173} =$	
1	$\dfrac{1}{\mathbf{i}^{173}} =$	
2	$\dfrac{1}{\mathbf{i}^{4*43+1}} =$	
3	$\dfrac{1}{\left(\mathbf{i}^4\right)^4 \mathbf{i}^1} =$	$\mathbf{i}^4 = 1$
4	$\dfrac{1}{(1)^4 \mathbf{i}^1} =$	
5	$\dfrac{1}{\mathbf{i}} =$	
6	$\dfrac{1}{\mathbf{i}} * \dfrac{\mathbf{i}}{\mathbf{i}} =$	
7	$\dfrac{\mathbf{i}}{\mathbf{i}^2} =$	
8	$\dfrac{\mathbf{i}}{-1} =$	
9	$-\mathbf{i}$	

Definitions: A **complex number** has the "standard" form

$$z = a + ib = (a, b) \text{ where } i = \sqrt{-1}$$
$$= (\text{Real Part}) + i(\text{Imaginary Part}) = (\text{Real Part}, \text{Imaginary Part})$$

so

$$"a" = \text{Real Part of } z$$
$$"b" = \text{Imaginary Part of } z$$

Also

$$\overline{z} = a - ib = \text{conjugate of } z \text{ [Change the SIGN of the Imag Part]}$$

&

$$|z| = \sqrt{a^2 + b^2} = \text{magnitude of } z$$

Note: Every real number "x" can be written as a complex number:

$$x = x + 0i$$
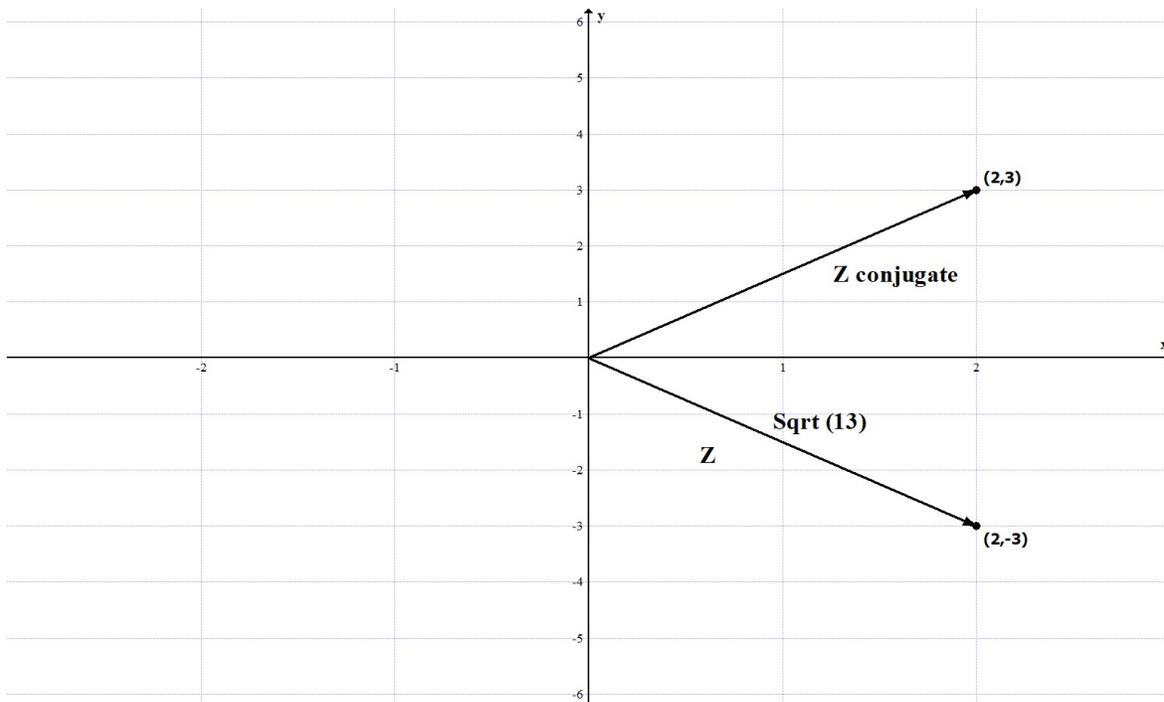
Example 4: Find the conjugate of $z_1 = 2 - 3i$
Solution:

The conjugate of $\mathbf{a + ib}$ is $\mathbf{a - ib} = 2 + 3\mathbf{i}$.

Example 5: Find the magnitude of $z_1 = 2 - 3i$.
Solution:

The magnitude of $\mathbf{a + ib}$ is $\sqrt{\mathbf{a^2 + b^2}} = \sqrt{(2)^2 + (3)^2} = \sqrt{13}$.

Since a complex number $\mathbf{z = a + ib}$ can be considered as the ordered pair $(2, -3)$, we can graph it using the Number Plane:

Note: We can think of a complex number as a **vector** since it has a magnitude and direction.

Operations with Complex Numbers:

Let

$$z_1 = a + ib = 2 + 3i$$
$$z_2 = c + id = -1 + 5i$$

Sum: $z_1 + z_2 = (a + ib) + (c + id) = \overbrace{(a + c)}^{Real\ Part} + \overbrace{(b + d)}^{Imag\ Part} i$

Example: $z_1 + z_2 = (2 + 3i) + (-1 + 5i) = 1 + 8i$

Difference: $z_1 - z_2 = (a + ib) - (c + id) = \overbrace{(a - c)}^{Real\ Part} + \overbrace{(b - d)}^{Imag\ Part} i$

Example: $z_1 - z_2 = (2 + 3i) - (-1 + 5i) = 3 - 2i$

Product: $z_1 * z_2 = (ac - bd) + (ad + bc)i$

Example:

$$z_1 * z_2 = (2 + 3i) * (-1 + 5i) = -2 + 10i - 3i + 15i^2$$
$$= -2 + 7i - 15 = -17 + 7i$$

Quotient: $\dfrac{z_1}{z_2} = \dfrac{a + ib}{c + id} = \dfrac{a + ib}{c + id} * \dfrac{c - id}{c - id} = \dfrac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$

Example:

$$\frac{z_1}{z_2} = \frac{2 + 3i}{-1 + 5i} = \frac{2 + 3i}{-1 + 5i} * \frac{-1 - 5i}{-1 - 5i} = \frac{(2 + 3i) * (-1 - 5i)}{(-1 + 5i) * (-1 - 5i)}$$

$$= \frac{-2 - 10i - 3i - 15i^2}{1 + 5i - 5i - 25i^2}$$

$$= \frac{13 - 13i}{26} = \frac{1}{2} - \frac{1}{2}i$$

Below are a few more examples in multiple choice format:

In **Examples 6-9,** define

$$\mathbf{z}_1 = 2 - 3\mathbf{i}$$
$$\mathbf{z}_2 = 5 + 4\mathbf{i}$$

Example 6: The real part of $\mathbf{z}_1 + \mathbf{z}_2$ equals

    A) -7
    B) -1
    C) 1
    D) 7

ANSWER: D

Solution:

Step	Equation	Reason
0	$\mathbf{z}_1 + \mathbf{z}_2 =$	
1	$(2 - 3\mathbf{i}) + (5 + 4\mathbf{i}) =$	
2	$(2 + 5) + (-3 + 4)\mathbf{i} =$	
3	$7 + \mathbf{i}$	

Example 7: The imaginary part of $z_1 - z_2$ equals

    A) -7
    B) -3
    C) 3
    D) 7

ANSWER: A

Solution:

Step	Equation	Reason
0	$z_1 - z_2 =$	
1	$(2-3\mathbf{i})-(5+4\mathbf{i}) =$	
2	$(2-5)+(-3-4)\mathbf{i} =$	
3	$-3-7\mathbf{i}$	

Example 8: The product $z_1 * z_2$ equals

    A) $-22-7\mathbf{i}$
    B) $-22+7\mathbf{i}$
    C) $22-7\mathbf{i}$
    D) $22+7\mathbf{i}$

ANSWER: C

Solution:

Step	Equation	Reason
0	$z_1 * z_2 =$	
1	$(2-3\mathbf{i})*(5+4\mathbf{i}) =$	
2	$10+8\mathbf{i}-15\mathbf{i}-12\mathbf{i}^2 =$	
3	$10+8\mathbf{i}-15\mathbf{i}-12(-1) =$	
4	$10+8\mathbf{i}-15\mathbf{i}+12 =$	
5	$22-7\mathbf{i}$	

Example 9: The quotient $\dfrac{\mathbf{z}_1}{\mathbf{z}_2}$ equals

A) $\dfrac{-2-23\mathbf{i}}{41}$

B) $\dfrac{-2+23\mathbf{i}}{41}$

C) $\dfrac{-2+23\mathbf{i}}{41}$

D) $\dfrac{2+23\mathbf{i}}{41}$

ANSWER: A

Solution:

Step	Equation	Reason
0	$\dfrac{\mathbf{z}_1}{\mathbf{z}_2}=$	
1	$\dfrac{2-3\mathbf{i}}{5+4\mathbf{i}}=$	
2	$\dfrac{2-3\mathbf{i}}{5+4\mathbf{i}}*\dfrac{5-4\mathbf{i}}{5-4\mathbf{i}}=$	
3	$\dfrac{10-8\mathbf{i}-15\mathbf{i}+12\mathbf{i}^2}{25-16\mathbf{i}^2}=$	
4	$\dfrac{10-8\mathbf{i}-15\mathbf{i}+12(-1)}{25-16(-1)}=$	
5	$\dfrac{10-8\mathbf{i}-15\mathbf{i}-12}{25+16}=$	
6	$\dfrac{-2-23\mathbf{i}}{41}=$	