Circles

[Center C(h, k) and Radius r]

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Your Personal Mathematics Trainer
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The **circle** is an important application of the distance formula.

Definition: A **circle** is the set of all points **P(x , y)** that are a fixed distance **r**, called the **radius**, from a fixed point **C(h ,k)**, called the **center.**

Circle Equation: The **equation of the circle** with **radius** $\mathbf{r}$ and **center** $\mathbf{C}(\mathbf{h},\mathbf{k})$, is given by

$$\sqrt{(\mathbf{x}-\mathbf{h})^2+(\mathbf{y}-\mathbf{k})^2}=\mathbf{r}\ ;\ (\mathbf{x}-\mathbf{h})^2+(\mathbf{y}-\mathbf{k})^2=\mathbf{r}^2$$

where **P(x , y)** represents a point on the circle.

Its **extreme points,** points on the graph that have a maximum/minimum x or y values, are

$$(\mathbf{h}-\mathbf{r},\mathbf{k})\ ;\ (\mathbf{h}+\mathbf{r},\mathbf{k})$$
$$(\mathbf{h},\mathbf{k}-\mathbf{r})\ ;\ (\mathbf{h},\mathbf{k}+\mathbf{r})$$

The **domain,** the projection of the graph onto the x-axis, is $[\mathbf{h}-\mathbf{r},\mathbf{h}+\mathbf{r}]_{\mathbf{x}}$. These are the *allowable* x values.

The **range**, the projection of the graph onto the y-axis, is $[\mathbf{k}-\mathbf{r},\mathbf{k}+\mathbf{r}]_{\mathbf{y}}$ These are the *allowable* y values.

The x coordinates of the **x-intercept points**, if any, are solutions of

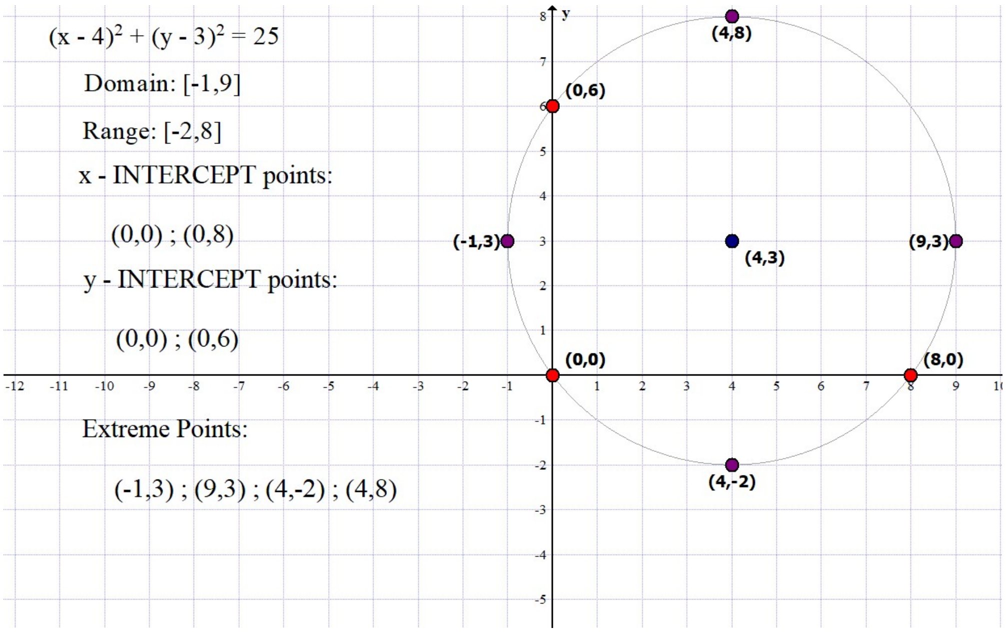
$(\mathbf{x}-\mathbf{h})^2+(0-\mathbf{k})^2=\mathbf{r}^2$ **Note:** y = 0

The y coordinates of the **y-intercept points**, if any, are solutions of

$(0-\mathbf{h})^2+(\mathbf{y}-\mathbf{k})^2=\mathbf{r}^2$ **Note:** x = 0

Note: These are the points where the graph of the circle intersects with either the x-axis or y-axis.

The following graph shows an example using these definitions:



Example 01: Given $\mathbf{C}(\mathbf{h},\mathbf{k}) = \mathbf{C}(1,-2)$ and radius $\mathbf{r} = 3$, find the following:

a. The **equation** of the circle:

Step	Equation	Reason
0	$(\mathbf{x}-\mathbf{h})^2 + (\mathbf{y}-\mathbf{k})^2 = \mathbf{r}^2$	
1	$(\mathbf{x}-[1])^2 + (\mathbf{y}-[-2])^2 = [3]^2$	Be careful with the minus signs
2	$(\mathbf{x}-1)^2 + (\mathbf{y}+2)^2 = 3^2$	
3	$\mathbf{x}^2 - 2\mathbf{x} + 1 + \mathbf{y}^2 + 4\mathbf{y} + 4 = 9$	
4	$\mathbf{x}^2 + \mathbf{y}^2 - 2\mathbf{x} + 4\mathbf{y} - 4 = 0$	Quadratic Form in x & y

b. The **extreme points**:

Step	Extreme Points	Reason
0	$(\mathbf{h}-\mathbf{r},\mathbf{k})$; $(\mathbf{h}+\mathbf{r},\mathbf{k})$	
1	Points: $(-2,-2)$; $(4,-2)$	
0	$(\mathbf{h},\mathbf{k}-\mathbf{r})$; $(\mathbf{h},\mathbf{k}+\mathbf{r})$	
1	Points: $(1,-5)$; $(1,1)$	

Note: The center $\mathbf{C}(1,-2)$ is the mid-point of the extreme points:

1. $(-2,-2)$; $(4,-2)$
2. $(1,-5)$; $(1,1)$

c. The **domain**: $[-2,4]$

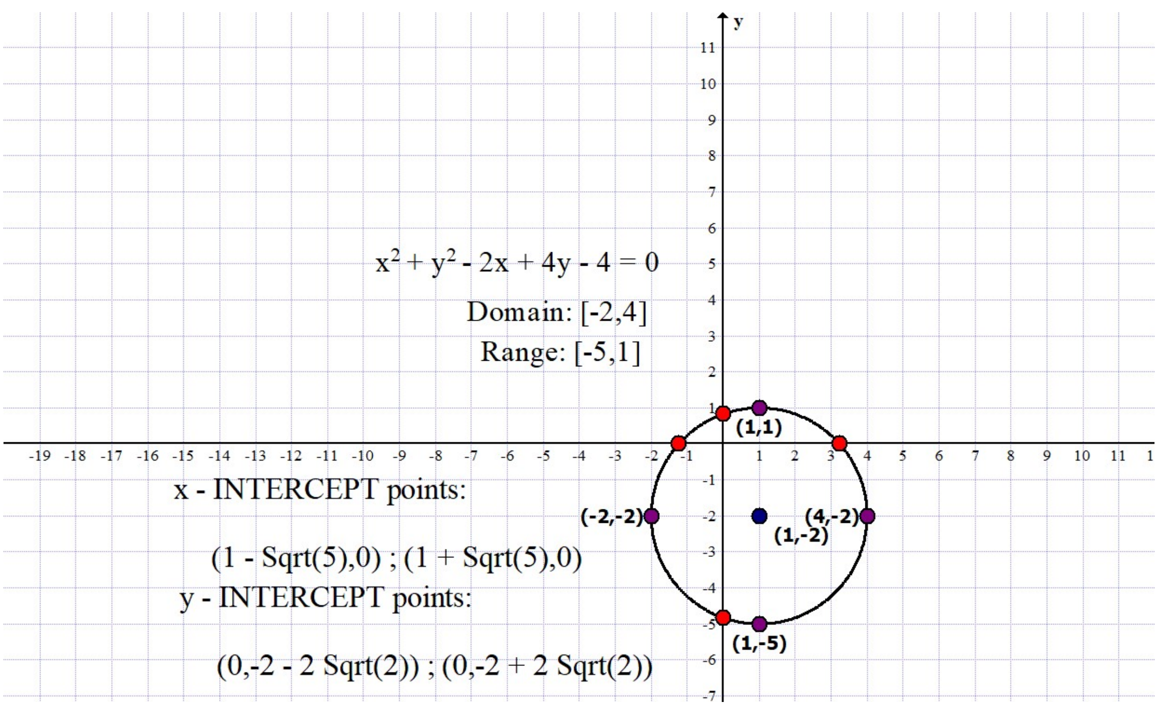
d. The **range**: $[-5,1]$

e. The **x-intercept points**: Set $\mathbf{y}=0$ and solve for x

Step	x-intercept points	Reason
0	$\mathbf{x}^2-2\mathbf{x}-4=0$	Solve
1	$\mathbf{x}=\frac{-\mathbf{b}\pm\sqrt{\mathbf{b}^2-4\mathbf{ac}}}{2\mathbf{a}}$; $\mathbf{a}=1$; $\mathbf{b}=-2$; $\mathbf{c}=-4$	
2	$\mathbf{x}=\frac{-[-2]\pm\sqrt{[-2]^2-4[1][-4]}}{2[1]}$	
3	$\mathbf{x}=\frac{2\pm\sqrt{4+16}}{2}$	
4	$\mathbf{x}=\frac{2\pm\sqrt{20}}{2}=\frac{2\pm2\sqrt{5}}{2}$	
5	$\mathbf{x}=1\pm\sqrt{5}$	
6	$\mathbf{x}=1-\sqrt{5}\approx-1.236$ || $\mathbf{x}=1+\sqrt{5}\approx3.236$	
7	Points: $\left(1-\sqrt{5},0\right)$; $\left(1+\sqrt{5},0\right)$ (-1.236, 0) ; (3.236, 0)	

The **y-intercept points**: Set $\mathbf{x} = 0$ and solve for y

Step	y-intercept points	Reason
0	$\mathbf{y}^2 + 4\mathbf{y} - 4 = 0$	Solve
1	$\mathbf{y} = \dfrac{-\mathbf{b} \pm \sqrt{\mathbf{b}^2 - 4\mathbf{ac}}}{2\mathbf{a}}$; $\mathbf{a} = 1$; $\mathbf{b} = 4$; $\mathbf{c} = -4$	
2	$\mathbf{y} = \dfrac{-[4] \pm \sqrt{[4]^2 - 4[1][-4]}}{2[1]}$	
3	$\mathbf{y} = \dfrac{-4 \pm \sqrt{16+16}}{2[1]}$	
4	$\mathbf{y} = \dfrac{-4 \pm \sqrt{32}}{2} = \dfrac{-4 \pm 4\sqrt{2}}{2}$	
5	$\mathbf{y} = -2 \pm 2\sqrt{2}$	
6	$\mathbf{y} = -2 - 2\sqrt{2} \approx -4.83$; $\mathbf{y} = -2 + 2\sqrt{2} \approx +0.83$	
7	Points: $(0, -2 - 2\sqrt{2})$; $\left(0, -2 + 2\sqrt{2}\right)$ (0, -4.828) ; (0, 828)	



Note: Always draw the graph and make sure that *all* points are where they are supposed to be. If NOT, "Fix it"!

Circles, ellipses, parabolas, and hyperbolas have the following form:

General Quadratic Equation in x & y:

$$\mathbf{A}\mathbf{x}^2 + \mathbf{B}\mathbf{x}\mathbf{y} + \mathbf{C}\mathbf{y}^2 + \mathbf{D}\mathbf{x} + \mathbf{E}\mathbf{y} + \mathbf{F} = 0$$

This will be a circle if $\mathbf{A} = \mathbf{C} \neq 0 \quad (\mathbf{B} = 0)$. We must *complete the square* twice, once for "x" and once for "y", to determine the Center & Radius.

Example 02: Given the circle defined by $\mathbf{x}^2 + \mathbf{y}^2 + 6\mathbf{x} - 10\mathbf{y} + 9 = 0$, find the following:

a. The **center** $\mathbf{C}(\mathbf{h},\mathbf{k})$ and **radius** $\mathbf{r}$:

Step	Equation	Reason
0	$\mathbf{x}^2 + \mathbf{y}^2 + 6\mathbf{x} - 10\mathbf{y} + 9 = 0$	A = 1 = C B = 0
1	$\mathbf{x}^2 + 6\mathbf{x} + [9] + \mathbf{y}^2 - 10\mathbf{y} + [25] = -9 + [9] + [25]$	$\left[\frac{1}{2}(6)\right]^2 = 9$ $\left[\frac{1}{2}(-10)\right]^2 = 25$
2	$(\mathbf{x}+3)^2 + (\mathbf{y}-5)^2 = 5^2$	
3	Center: $\mathbf{C}(-3,5)$; Radius: $\mathbf{r} = 5$	

b. The **extreme points**:

Step	Extreme Points	Reason
0	$(\mathbf{h}-\mathbf{r},\mathbf{k})$; $(\mathbf{h}+\mathbf{r},\mathbf{k})$	
1	Points: $(-8,5)$; $(2,5)$	
0	$(\mathbf{h},\mathbf{k}-\mathbf{r})$; $(\mathbf{h},\mathbf{k}+\mathbf{r})$	
1	Points: $(-3,0)$; $(-3,10)$	

c. The **domain**: $[-8,2]$

d. The **range**: $[0,10]$

e. The **x-intercept points**: $\mathbf{y} = 0$

Step	x-intercept points	Reason
0	$x^2+6x+9=0$	Solve
1	$(x+3)^2=0$	
2	$x+3=0$ $x=-3$	
3	Point: $(-3,0)$	

f. The **y-intercept points**: $x=0$

Step	y-intercept points	Reason
0	$y^2-10y+9=0$	Solve
1	$(y-1)(y-9)=0$	
2	$y-1=0$ ‖ $y-9=0$ $y=1$ ‖ $y=9$	
3	Points: $(0,1)$; $(0,9)$	

Graph: