FUNctions: Continuity

[Continuity – NO breaks in the Graph]

[Discontinuity – breaks in the Graph]

Hole
Finite Jump
Vertical Asymptote

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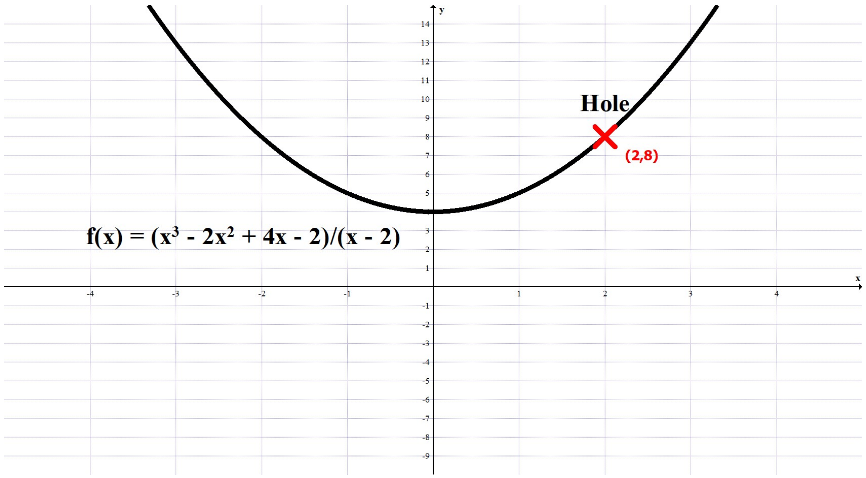
If a function does NOT have *any* breaks in its graph, it is said to be a **continuous function**. There are three (3) types of breaks (called **discontinuities**) our graphs may have:

1. Hole: $\mathbf{y} = \mathbf{f}(\mathbf{x}) = \dfrac{\mathbf{x}^3 - 2\mathbf{x}^2 + 4\mathbf{x} - 8}{\mathbf{x} - 2}$ $\left[= \mathbf{x}^2 + 4 \; ; \; \mathbf{x} \neq 2\right]$

The $\mathbf{Dom\ f} = \mathbb{R}_{\mathbf{x}} \setminus \{2\}$ and we denote the hole in the graph at (2,8) by an "x":

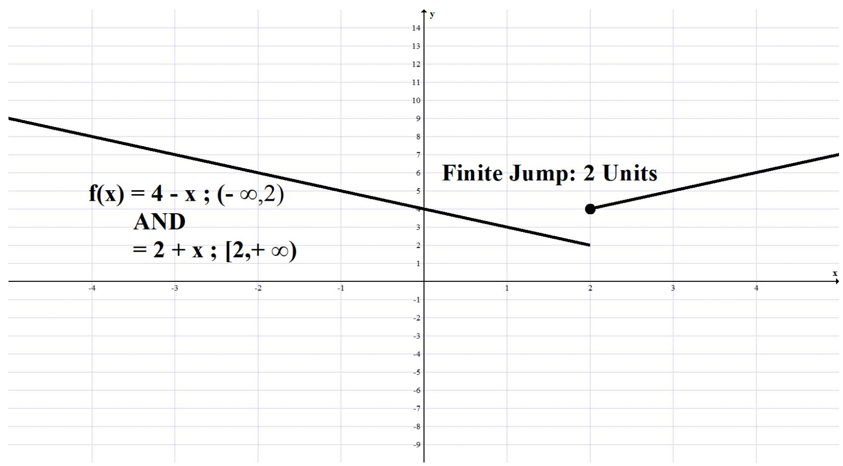


Note: " Although the *entire* graph was drawn, our analysis just gives us a small portion of the graph "around the hole" We will also show the *entire* graphs in the examples below.

2. Finite Jump: $\mathbf{y} = \mathbf{f}(\mathbf{x}) = \begin{cases} 4 - \mathbf{x} & \text{if } \mathbf{x} \in (-\infty, 2) \\ 2 + \mathbf{x} & \text{if } \mathbf{x} \in [2, +\infty) \end{cases}$

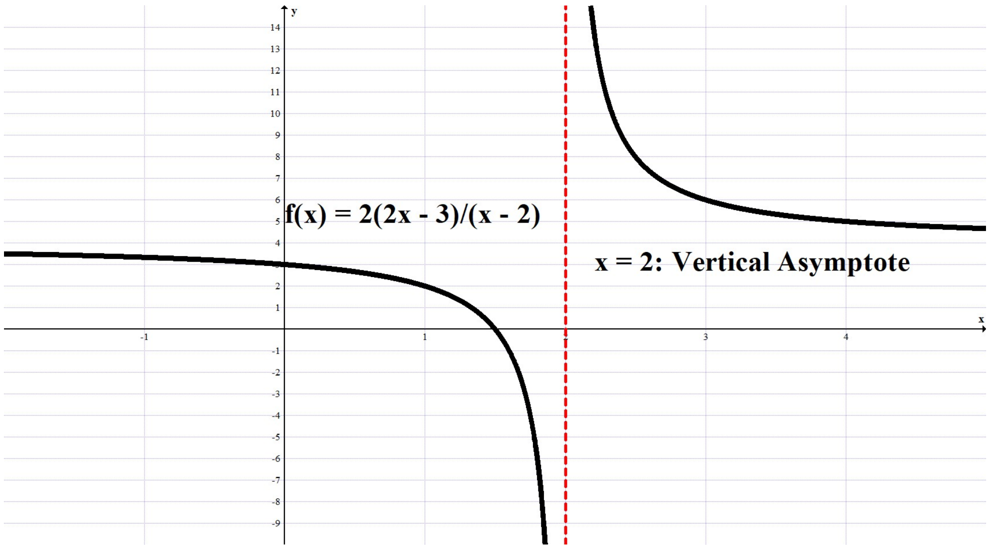
   Note: $\mathbf{Dom\,f} = (-\infty, 2) \cup [2, +\infty) = (-\infty, +\infty)_{\mathbf{x}} = \mathbb{R}_{\mathbf{x}}$

   The graph has a finite jump of 2 units at x = 2:



3. Vertical asymptote: $\mathbf{f}(\mathbf{x}) = \frac{2(2\mathbf{x}-3)}{\mathbf{x}-2}$

   The vertical line **x = 2** is called a **vertical asymptote** of the function **f**. Note that as the x-values get "closer and closer" to x = 2, the corresponding f(x) values get increase without bound (we say f(x) "goes to positive infinity": $\mathbf{f}(\mathbf{x}) \to +\infty$) on one side of **x = 2** (*right)* and decrease without bound (we say f(x) "goes to negative infinity": $\mathbf{f}(\mathbf{x}) \to -\infty$) on the other side of **x = 2** (*left*):



Theorem (Fundamental Theorem of Continuity): Let $\mathbf{f}$ be a function defined on an interval $\mathbf{I}$. Assume that

1. $\mathbf{f}$ is continuous for all $\mathbf{x} \in \mathbf{I}$ (NO breaks – no holes, finite jumps or vertical asymptotes)
2. $\mathbf{f}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in \mathbf{I}$ (NO x-intercept points)

Then either

- $\mathbf{f}(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbf{I}$ (*always positive)*

or

- $\mathbf{f}(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathbf{I}$ (*always negative*)

Note:

1. **Pos f** will represent the x-axis regions where $\mathbf{f}(\mathbf{x}) > 0$
2. **Neg f** will represent the x-axis regions where $\mathbf{f}(\mathbf{x}) < 0$

Example: Given $\mathbf{f}(\mathbf{x}) = \mathbf{x}^3 - 3\mathbf{x}$, find where it is negative, zero, and positive.

Solution:

We first find the x-intercept points:

$$\mathbf{f}(\mathbf{x}) = \mathbf{x}^3 - 3\mathbf{x} = \mathbf{x}\left(\mathbf{x}^2 - 3\right) \overset{\text{SET}}{=} 0 \Rightarrow \mathbf{x} = 0, \pm\sqrt{3} \Rightarrow \left(-\sqrt{3},0\right),(0,0),\left(\sqrt{3},0\right)$$

These three (3) points divide the x-axis into *four* intervals:

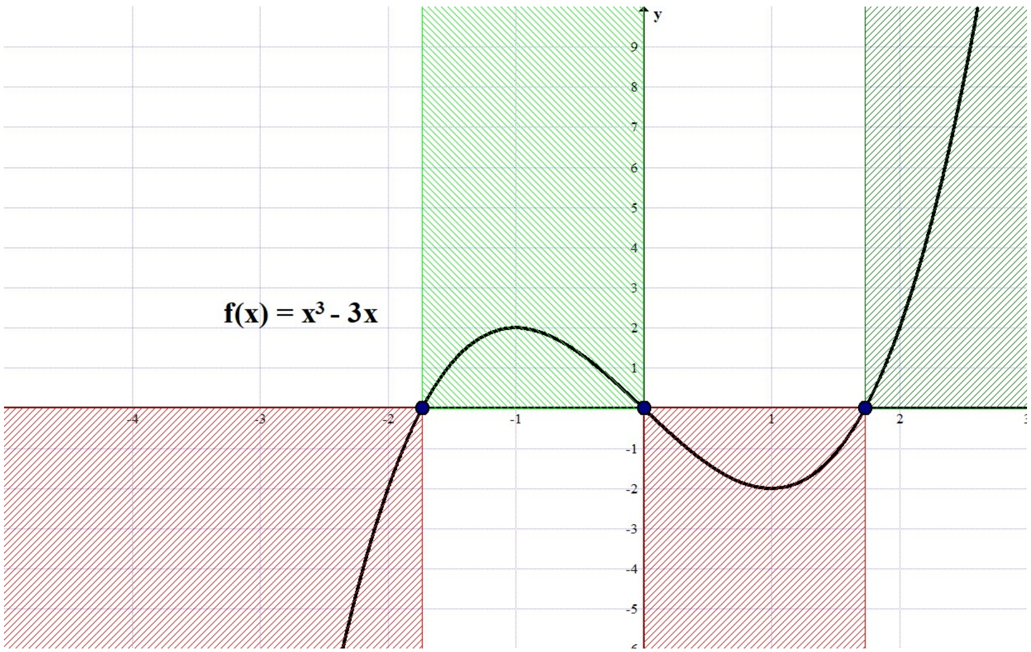
$$\left(-\infty,-\sqrt{3}\right),\left(-\sqrt{3},0\right),\left(0,\sqrt{3}\right),\left(\sqrt{3},+\infty\right)$$

Each of these intervals satisfies the hypothesis in the above theorem so we can select a representative point in each interval to determine the sign in the *entire* interval:

$$\underset{\mathbf{f}(-2)<0}{\overbrace{\_\_\_\_\_\_\_\_\_\_}^{-}}\ -\sqrt{3}\ \underset{\mathbf{f}(-1)>0}{\overbrace{\_\_\_\_\_\_}^{+}}\ 0\ \underset{\mathbf{f}(1)<0}{\overbrace{\_\_\_\_\_\_}^{-}}\ \sqrt{3}\ \underset{\mathbf{f}(2)>0}{\overbrace{\_\_\_\_\_\_\_\_\_\_}^{+}}$$

Therefore, $\textbf{Pos f} = \left(-\sqrt{3},0\right) \cup \left(\sqrt{3},+\infty\right)_{\mathbf{x}}$ and $\textbf{Neg f} = \left(-\infty,-\sqrt{3}\right) \cup \left(0,\sqrt{3}\right)_{\mathbf{x}}$

The function is positive $(\mathbf{f}(\mathbf{x}) > 0)$ for regions above the x-axis and negative $(\mathbf{f}(\mathbf{x}) < 0)$ in the regions below the x-axis:



f(x) = x³ - 3x
y
9
8
7
6
5
4
3
2
1
-4
-3
-2
-1
1
2
-1
-2
-3
-4
-5

