Functions
Exponential

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An *exponential function* has the form

$$\mathbf{f(x) = b^x \ ; \ 0 < b < 1 \text{ or } b > 1}$$

Note: The restrictions on the base are because we are only considering real valued functions.

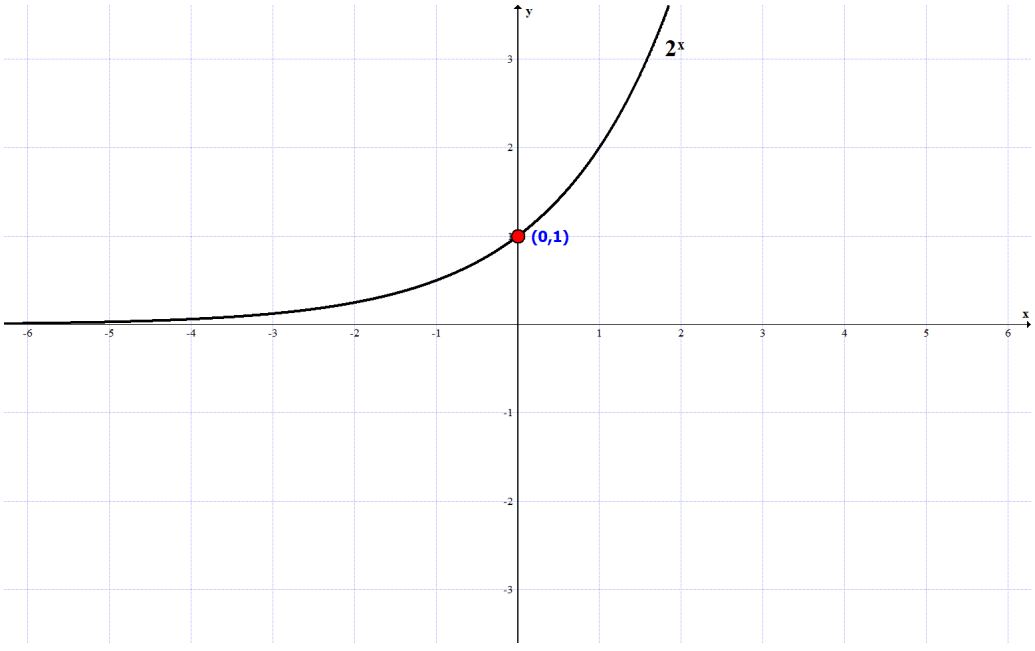
Note:

1. **Power function:** $\mathbf{f(x) = x^n} \ ; [\text{Variable}]^{\text{Constant}}$
2. **Exponential function:** $\mathbf{f(x) = b^x} \ ; [\text{Constant}]^{\text{Variable}}$

There are two (2) shapes an exponential function can have:

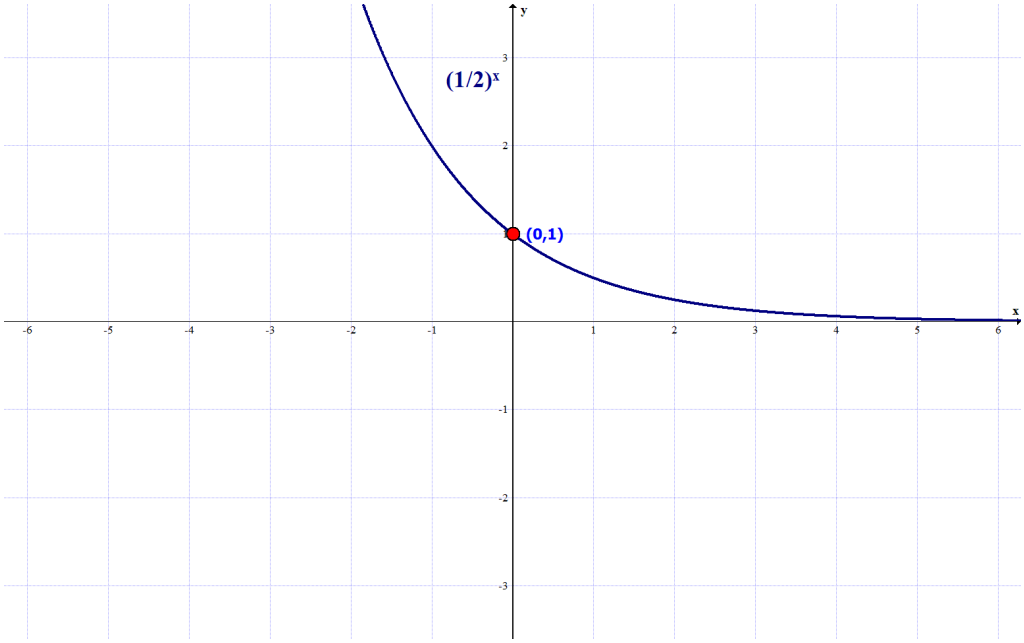
1. $\mathbf{b > 1}$**:**

   When the base satisfies $\mathbf{b > 1}$, the graph has the shape of $\mathbf{f(x) = 2^x}$:

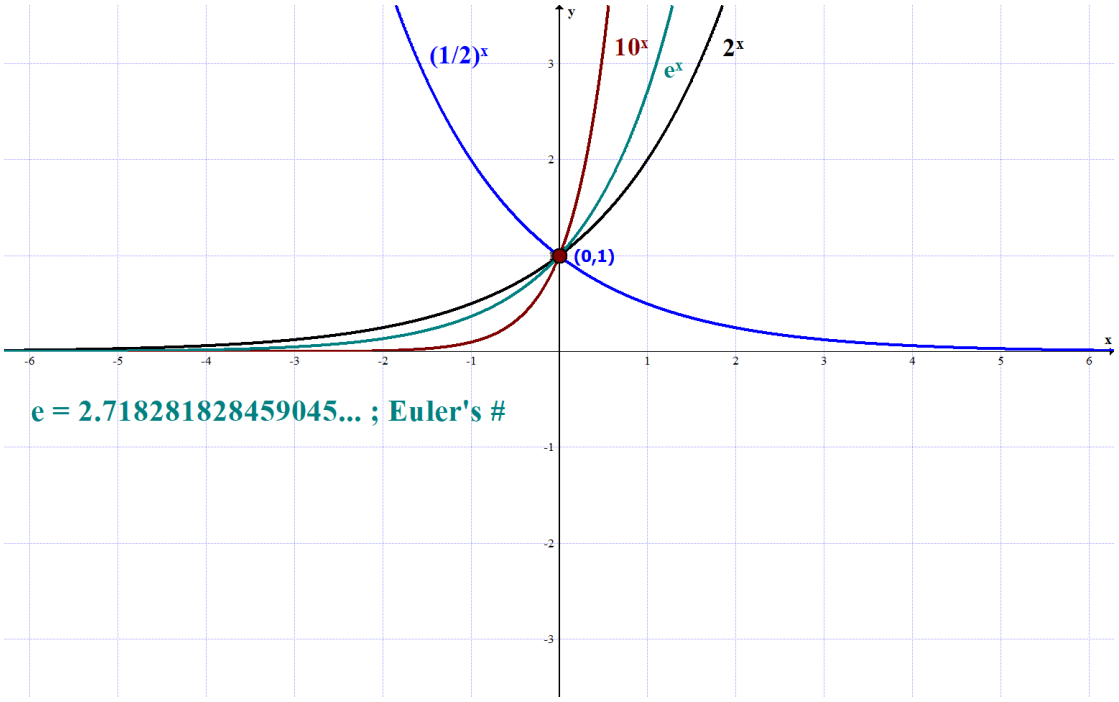


2. $b < 1$:

When the base satisfies $0 < b < 1$, the graph has the shape of $f(x) = \left(\frac{1}{2}\right)^x$:



Here are some graphs of the important exponential functions:



Note:

1. **The *domain* of every exponential function is the set of real numbers: $\mathbb{R}_x$. The *range* of every exponential function is the set of real numbers: $\mathbb{R}_y$**
2. **All exponential functions pass through $(0,1)$ since $b^0 = 1$.**
3. **They all have $y = 0$ as a horizontal asymptote.**

Since each exponential function $f(x) = b^x$ is one to one (1-1), it has an *inverse function* called a logarithmic function, denoted $f(x) = \log_b(x)$. We will consider them next. As the table below shows, for each exponential property, there is a corresponding logarithmic property:

	Exponential	Logarithmic
	(One Base)	
1	$b^0 = 1$	$\log_b 1 = 0$
2	$b^1 = b$	$\log_b b = 1$
3	$b^x = b^y \Leftrightarrow x = y$ (1-1)	$\log_b x = \log_b y \Leftrightarrow x = y$ (1-1)
4	$\log_b b^x = x$ (Inverse)	$b^{\log_b x} = x$ (Inverse)
5	$b^x = y \Leftrightarrow x = \log_b y$ Trade exponential for logarithmic	$x = \log_b y \Leftrightarrow b^x = y$ Trade logarithmic for exponential
6	$(b^x)^y = b^{x*y}$	$\log_b x^y = y * \log_b x$ Trade exponential for multiplication
7≠	$(b^x)^y \neq y * b^x$	$(\log_b x)^y \neq y * \log_b x$
8	$b^x * b^y = b^{x+y}$	$\log_b (x * y) = \log_b x + \log_b y$ Trade multiplication for addition
9≠	$b^{x+y} \neq b^x + b^y$	$\log_b (x + y) \neq \log_b x + \log_b y$
10≠	$b^{x-y} \neq b^x - b^y$	$\log_b (x - y) \neq \log_b x - \log_b y$

11	$\frac{b^x}{b^y} = b^{x-y} = \frac{1}{b^{y-x}}$	$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$ **Trade division for subtraction**
12 $\neq$	$\frac{b^x}{b^y} \neq \frac{x}{y}$	$\frac{\log_b x}{\log_b y} \neq \frac{x}{y}$
13		$\log_b x = \frac{\log_{10} x}{\log_{10} b} = \frac{\log_e x}{\log_e b}$ **Change of Base**
	(Two Bases)	
14	$(a*b)^x = a^x * b^x$	
15	$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$	
16	$b^{-x} = \frac{1}{b^x}\ ;\ \frac{1}{b^{-x}} = b^x$	
17	$\left(\frac{a}{b}\right)^{-x} = \left(\frac{b}{a}\right)^x$	
18 $\neq$	$(a+b)^x \neq a^x + b^x$	
19 $\neq$	$(a-b)^x \neq a^x - b^x$	
20	$a^x b^y = a^x b^y$	

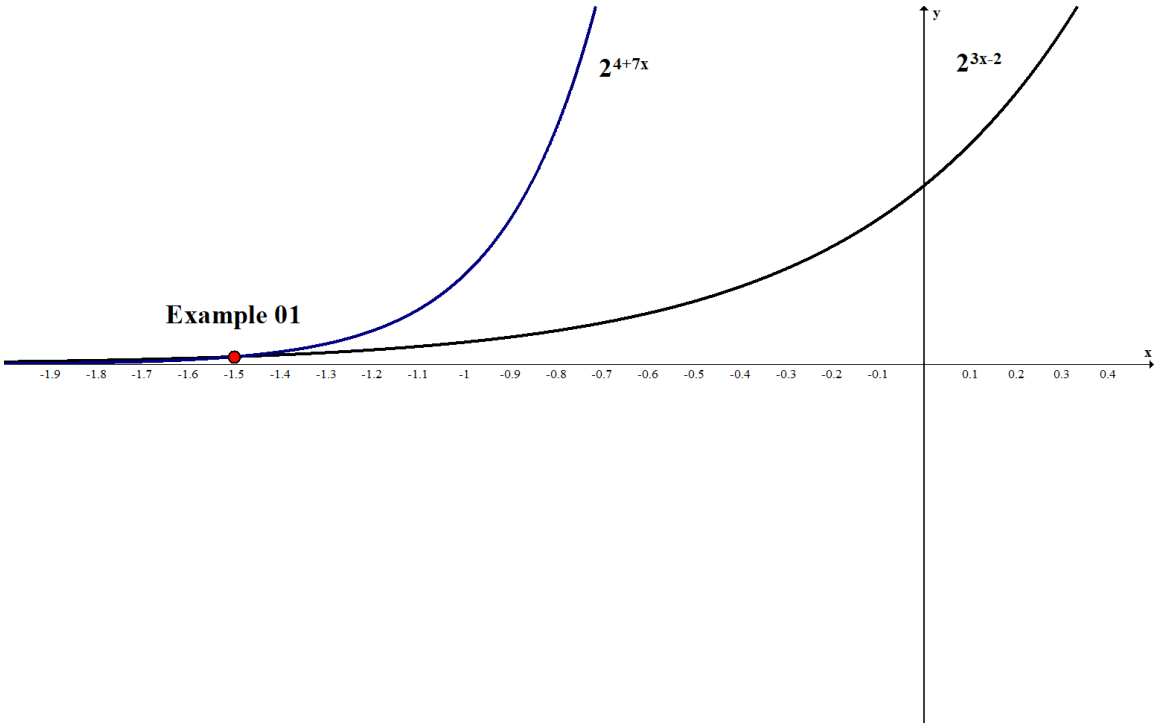
We use these properties, among other things, to solve exponential equations in these notes and logarithmic equations in forthcoming notes:

Example 01:

$2^{3\mathbf{x}-2} = 2^{4+7\mathbf{x}} \Rightarrow \mathbf{x} = ?$

$\text{Property Exp 03} \Rightarrow 3\mathbf{x} - 2 = 4 + 7\mathbf{x} \Rightarrow -6 = 4\mathbf{x} \Rightarrow \mathbf{x} = -\frac{3}{2}$

Visual Solution:



Check: $2^{3*(-3/2)-2} = 2^{4+7*(-3/2)} = \frac{1}{64\sqrt{2}} \approx 0.11045$

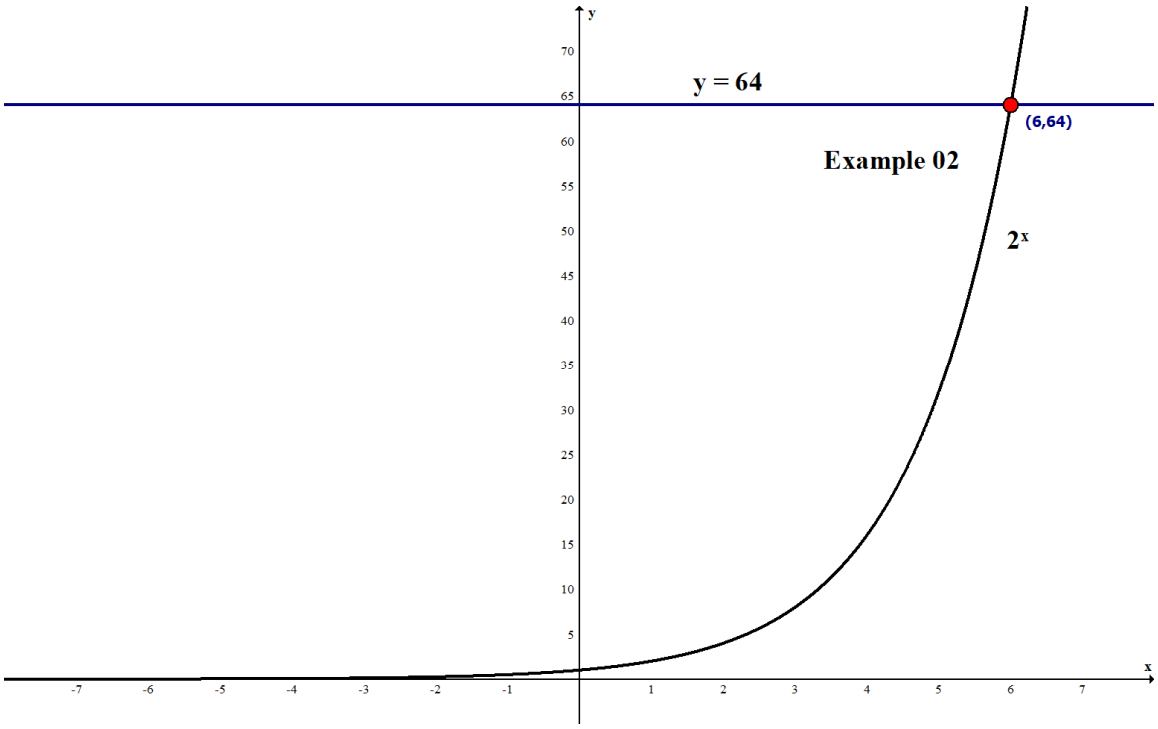
Example 02:

$2^{x} = 64 \Rightarrow \mathbf{x} = ?$

$2^{x} = 64 = 2^{6}$

Property Exp 03 $\Rightarrow \mathbf{x} = 6$

Visual Solution:



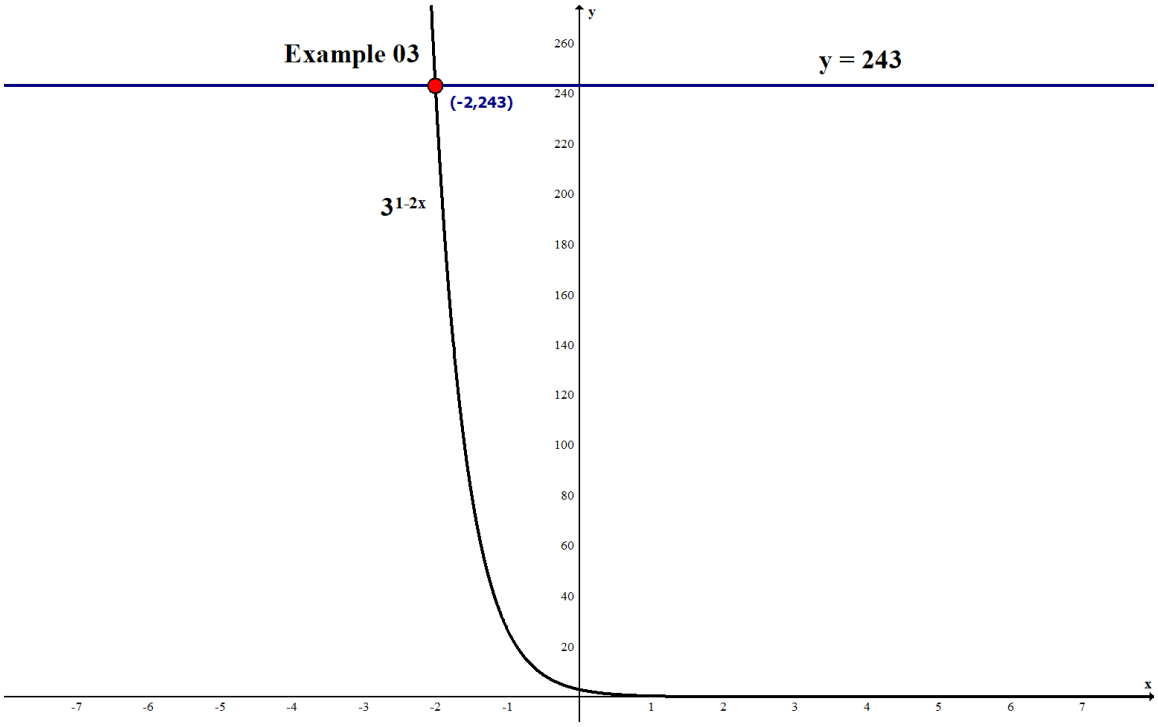
Example 03:

$3^{1-2\mathbf{x}} = 243 \Rightarrow \mathbf{x} = ?$

$3^{1-2\mathbf{x}} = 243 = 3^5$

Property Exp 03 $\Rightarrow 1 - 2\mathbf{x} = 5 \Rightarrow -2\mathbf{x} = 4 \Rightarrow \mathbf{x} = -2$

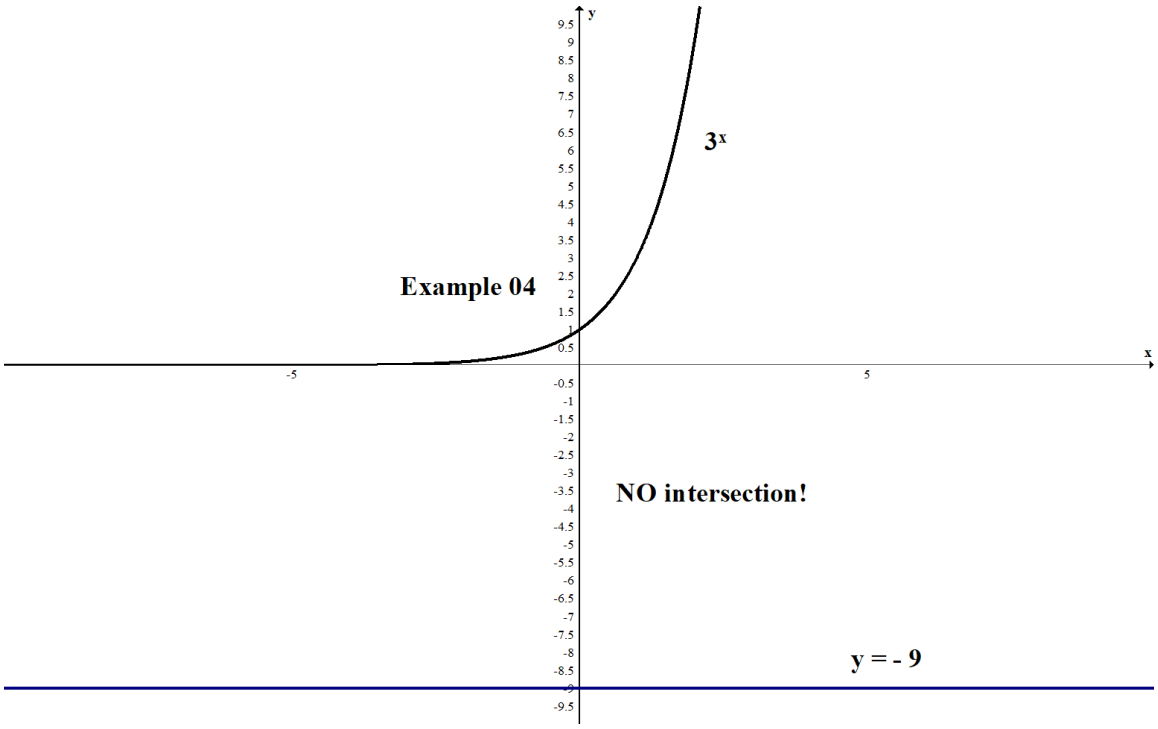
Visual Solution:



Example 04:

$3^{x} = -9 \Rightarrow \mathbf{x} = \nexists$; Trash!

Visual Solution:



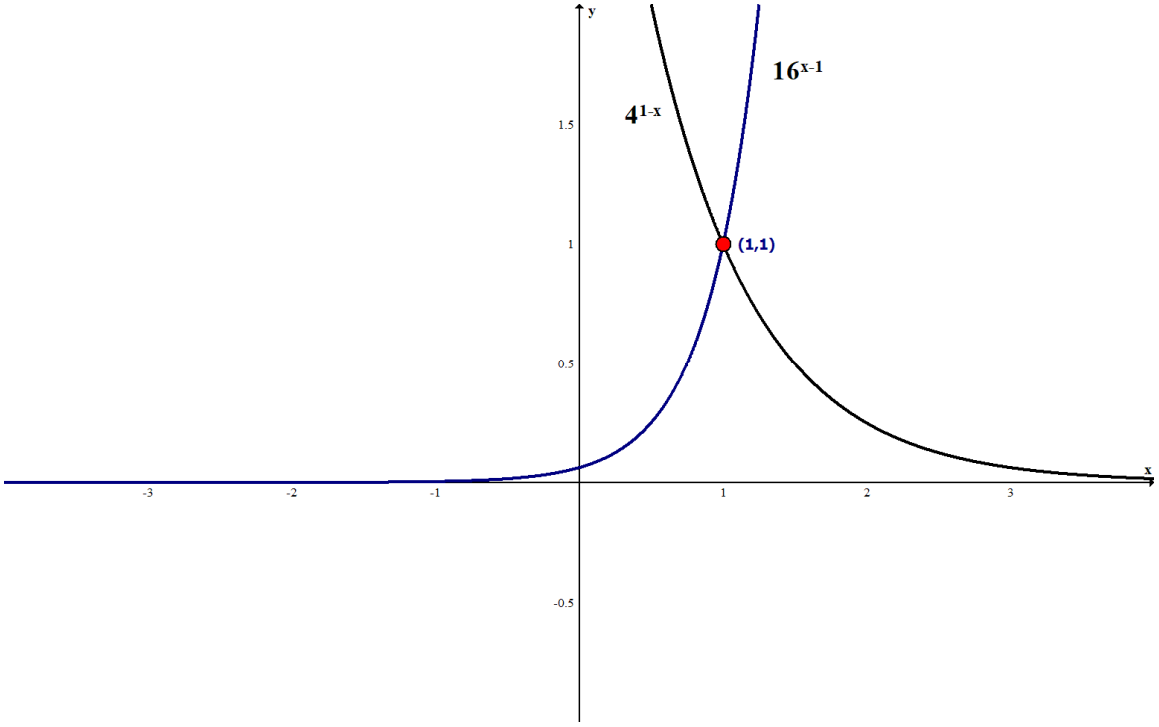
Example 05 :

$4^{1-x} = 16^{x-1} \Rightarrow \mathbf{x} = ?$

Property Exp 06 $\Rightarrow 4^{1-x} = \left(4^2\right)^{x-1} = 4^{2(x-1)} = 4^{2x-2}$

$\Rightarrow 1 - \mathbf{x} = 2\mathbf{x} - 2 \Rightarrow 3 = 3\mathbf{x} \Rightarrow \mathbf{x} = 1$

Visual Solution:



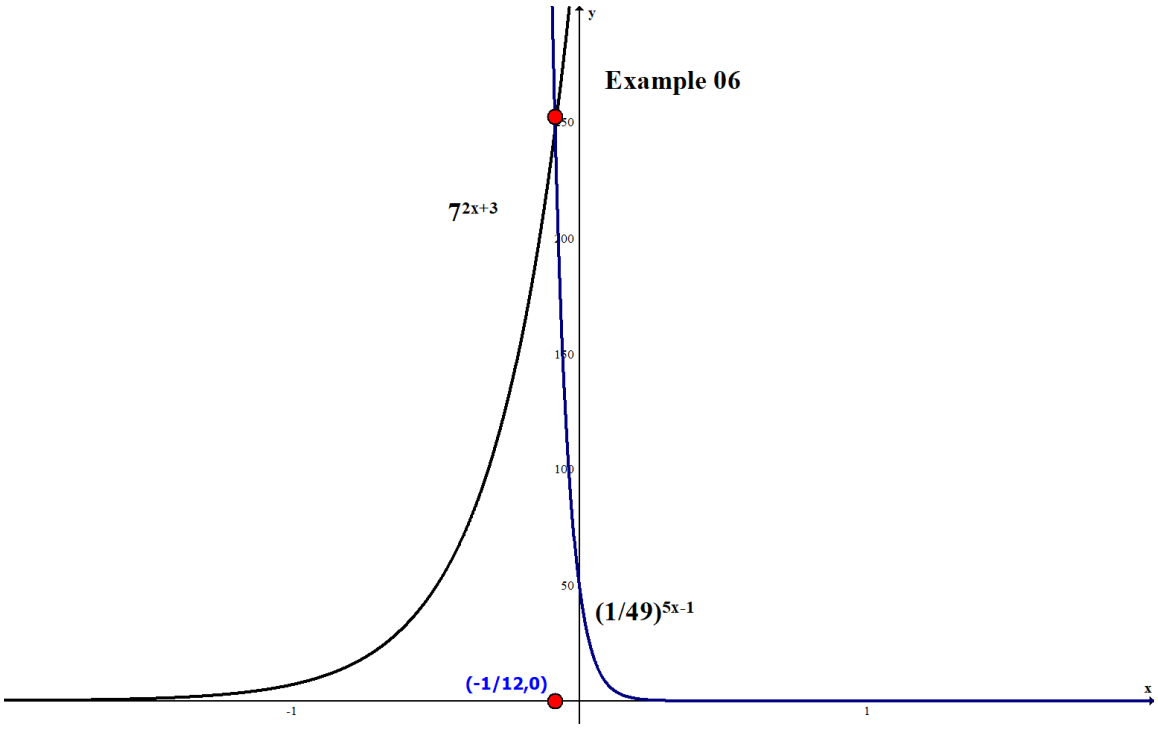
Example 06:

$$7^{2\mathbf{x}+3} = \left(\frac{1}{49}\right)^{5\mathbf{x}-1} \Rightarrow \mathbf{x} = ?$$

$$\text{Property Exp 11 \& 06} \Rightarrow 7^{2\mathbf{x}+3} = \left(7^{-2}\right)^{5\mathbf{x}-1} = 7^{-2(5\mathbf{x}-1)} = 7^{-10\mathbf{x}+2}$$

$$\text{Property Exp 03} \Rightarrow 2\mathbf{x}+3 = -10\mathbf{x}+2 \Rightarrow 12\mathbf{x} = -1 \Rightarrow \mathbf{x} = -\frac{1}{12}$$

Visual Solution:



Check: $7^{2*(-1/12)+3} = \left(\frac{1}{49}\right)^{5*(-1/12)-1} \approx 247.996$

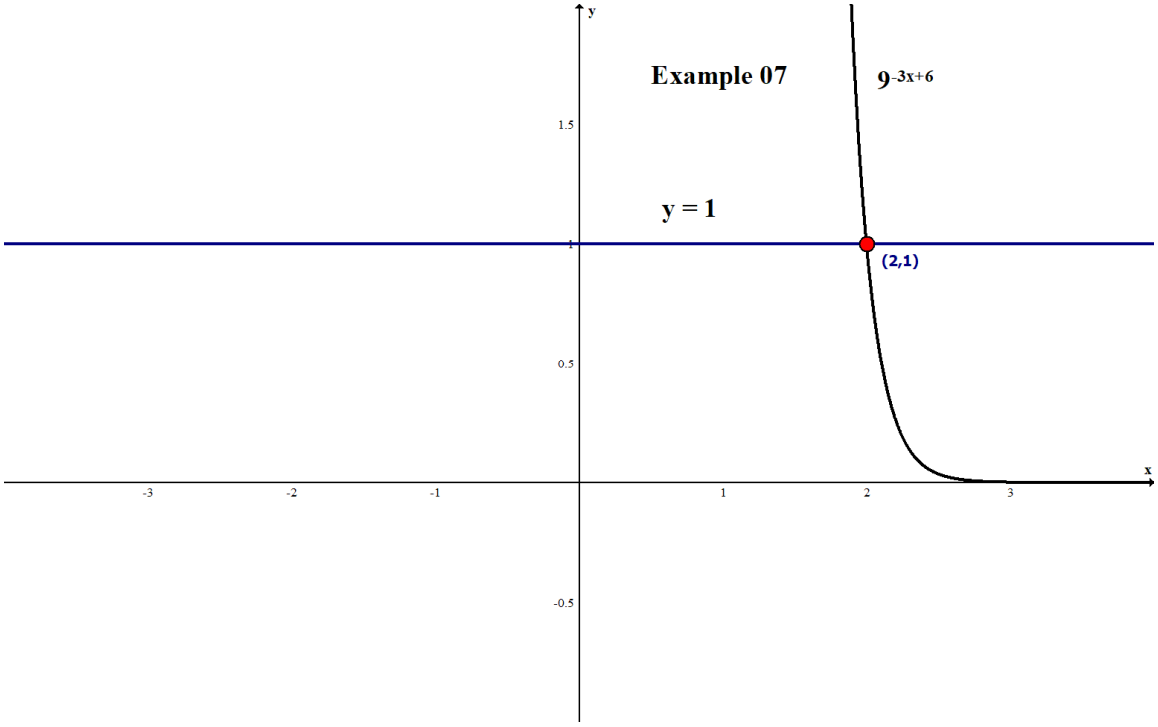
Example 07:

$9^{-3x+6} = 1 \Rightarrow \mathbf{x} = ?$

Property Exp 01 $\Rightarrow 9^{-3x+6} = 1 = 9^0$

Property Exp 03 $\Rightarrow -3\mathbf{x} + 6 = 0 \Rightarrow 3\mathbf{x} = 6 \Rightarrow \mathbf{x} = 2$

Visual Solution:



Example 08:

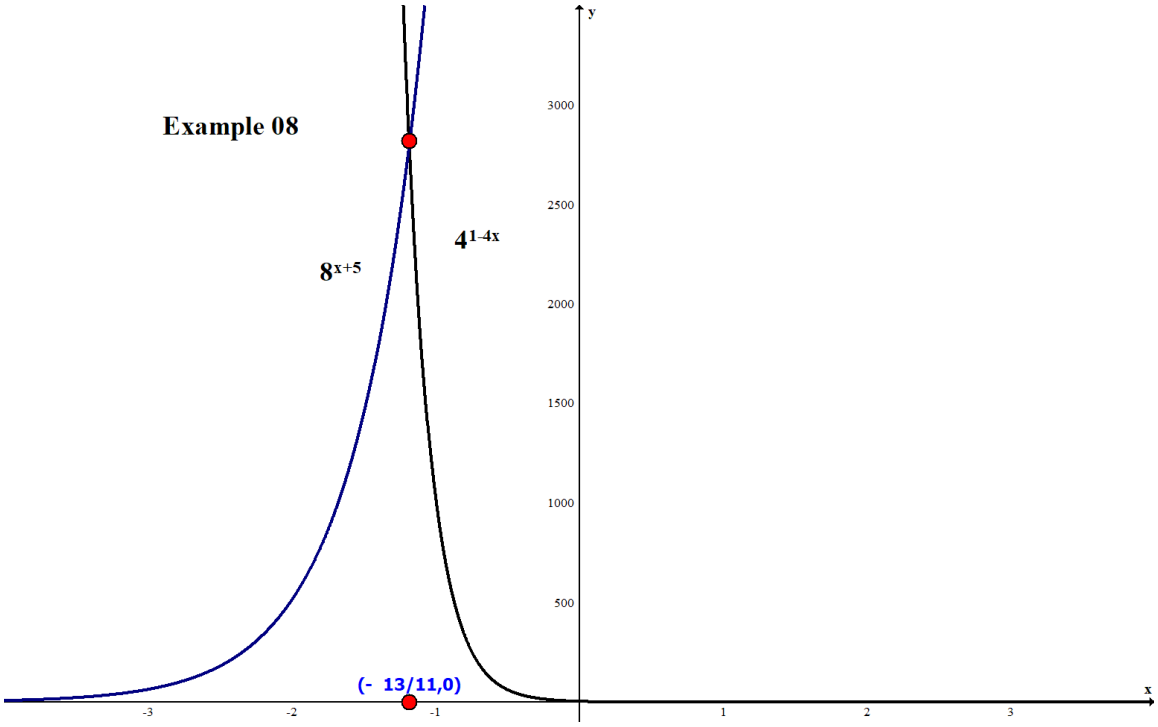
$4^{1-4\mathbf{x}} = 8^{\mathbf{x}+5} \Rightarrow \mathbf{x} = ?$

Property Exp 06 $\Rightarrow \left(2^2\right)^{1-4\mathbf{x}} = \left(2^3\right)^{\mathbf{x}+5} \Rightarrow 2^{2(1-4\mathbf{x})} = 2^{3(\mathbf{x}+5)}$

Property 03 $\Rightarrow 2(1-4\mathbf{x}) = 3(\mathbf{x}+5) \Rightarrow 2-8\mathbf{x} = 3\mathbf{x}+15$

$\Rightarrow -11\mathbf{x} = 13 \Rightarrow \mathbf{x} = -\dfrac{13}{11}$

Visual Solution:



Check: $4^{1-4*(-13/11)} = 8^{(-13/11)+5} \approx 2806.479$

Example 09:

$2^{x^2+2x} = 4$

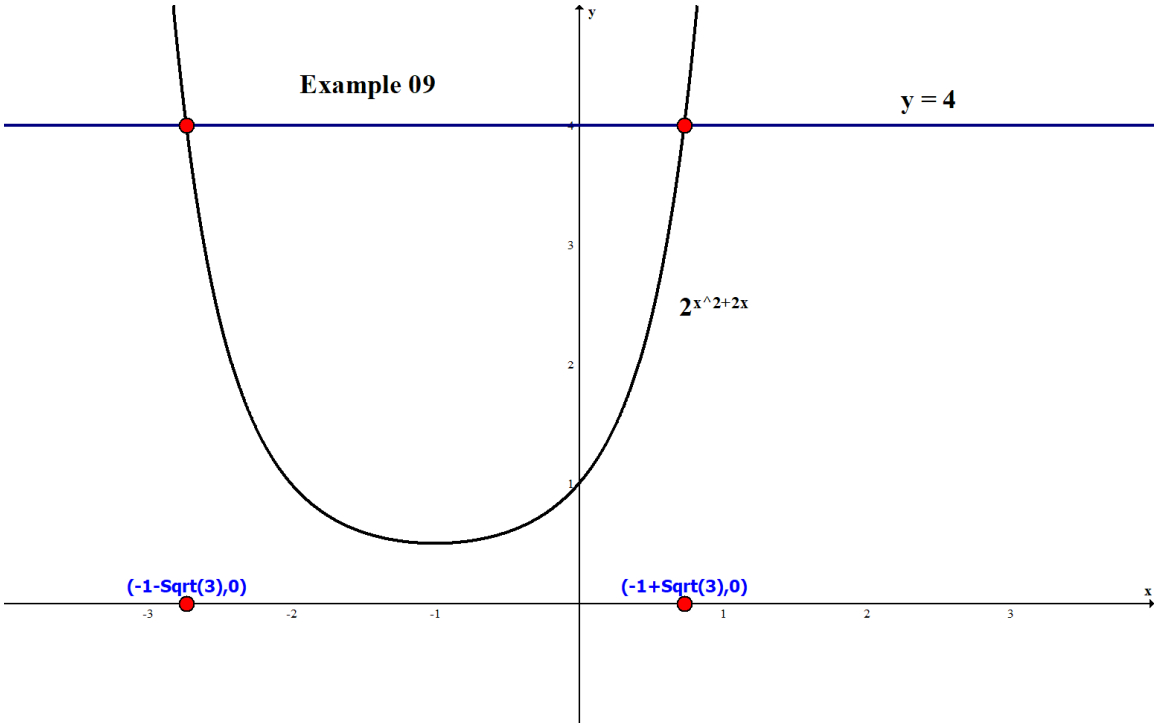
$\Rightarrow 2^{x^2+2x} = 2^2$

Property Exp 03 $\Rightarrow x^2 + 2x = 2 \Rightarrow x^2 + 2x - 2 = 0$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-2)}}{2(1)} = \frac{-2 \pm \sqrt{4+8}}{2} = \frac{-2 \pm \sqrt{12}}{2} = \frac{-2 \pm 2\sqrt{3}}{2}$$

$$= -1 \pm \sqrt{3} = \begin{cases} -1 - \sqrt{3} \approx -2.732 \\ -1 + \sqrt{3} \approx 0.732 \end{cases}$$

Visual Solution:



Check: $2^{\left(-1-\sqrt{3}\right)^2+2*\left(-1-\sqrt{3}\right)} = 4$; $2^{\left(-1+\sqrt{3}\right)^2+2*\left(-1+\sqrt{3}\right)} = 4$

Example 10:

$$2^{\mathbf{x}^2+2\mathbf{x}} = \frac{1}{32}$$

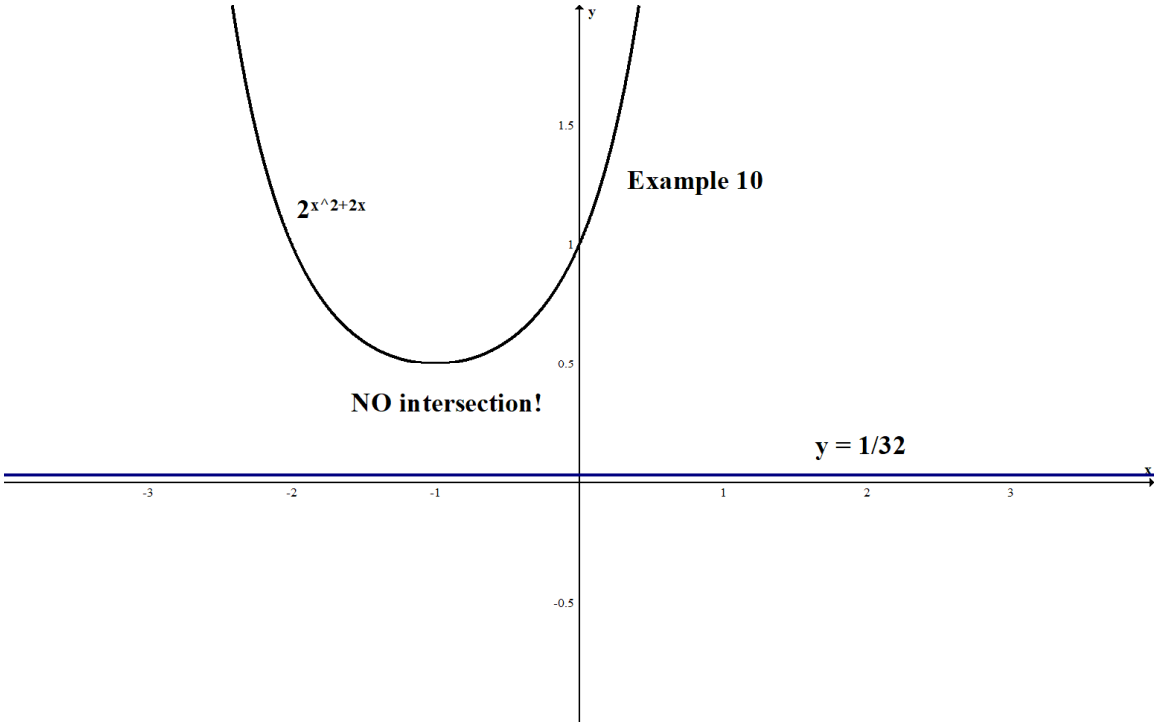
Property Exp 11 $\Rightarrow 2^{\mathbf{x}^2+2\mathbf{x}} = 2^{-5}$

Property Exp 03 $\Rightarrow \mathbf{x}^2 + 2\mathbf{x} = -5 \Rightarrow \mathbf{x}^2 + 2\mathbf{x} + 5 = 0$

$$\mathbf{x} = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4\mathbf{i}}{2}$$

$$= -1 \pm 2\mathbf{i} = \begin{cases} -1 - 2\mathbf{i} \\ -1 + 2\mathbf{i} \end{cases}$$

Visual Solution:



Check: $2^{(-1-\mathbf{i})^2+2*(-1-\mathbf{i})} = 4$; $2^{(-1+\mathbf{i})^2+2*(-1+\mathbf{i})} = 4$

The following two (2) exponential equations require the logarithmic properties to solve which we will solve in the Logarithmic Notes:

1. $5^{3x-4} = 17$

2. $3^{2-3x} = 7^{2x-11}$

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