FUNctions
Inverse of a FUNction

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Your Personal Mathematics Trainer
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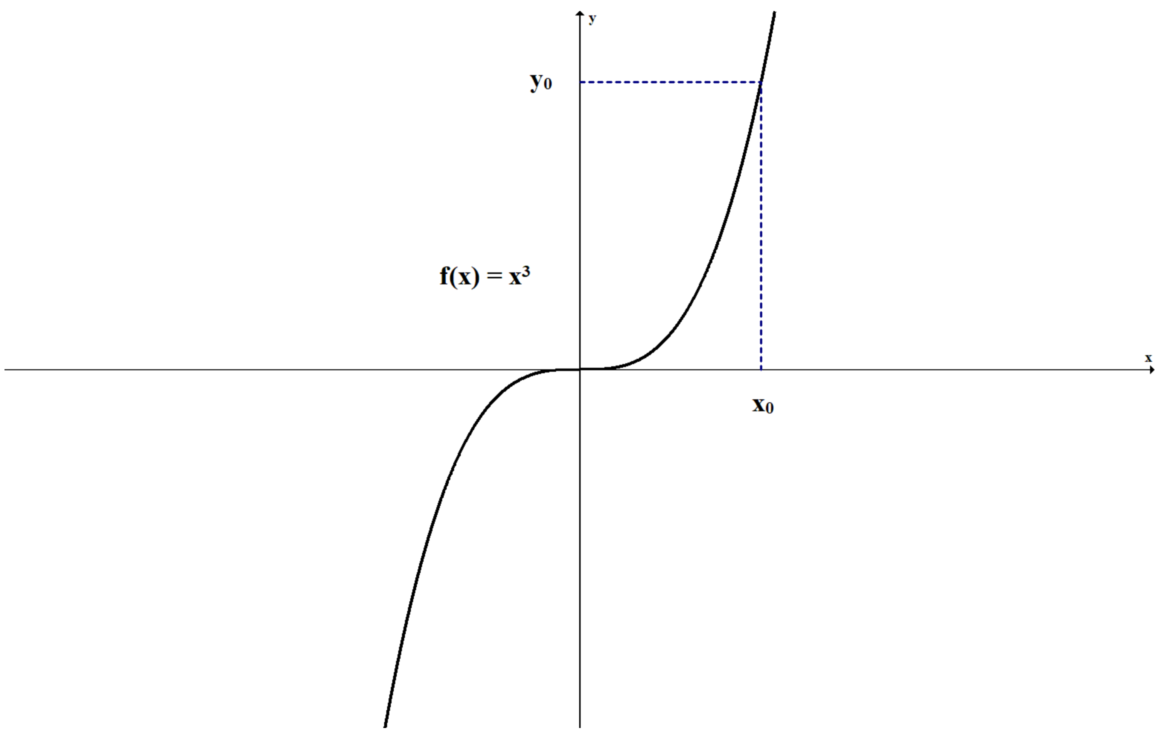
Consider $f(x) = x^3$; Dom $f = \mathbb{R}_x$; Rng $f = \mathbb{R}_y$:



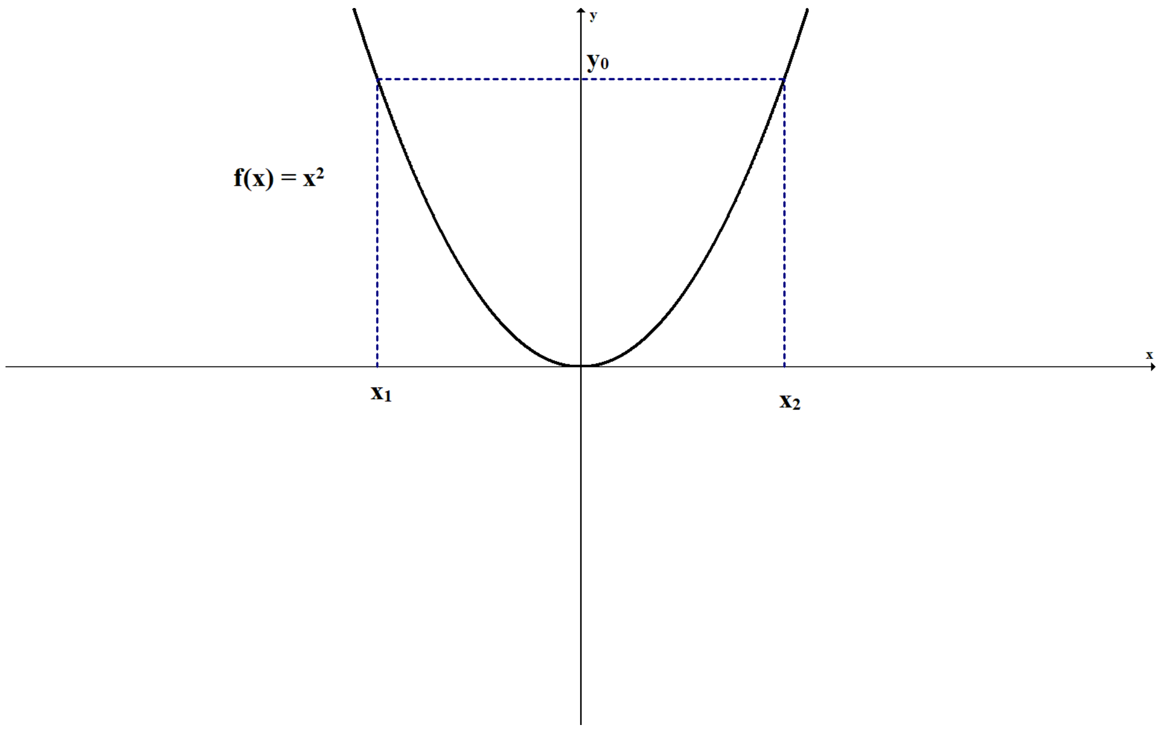
Let $y_0 \in$ Rng f. Setting $f(x) = x^3 \overset{SET}{=} y_0 \Rightarrow x \equiv x_0 = \sqrt[3]{y_0}$. Thus, there exists a *unique* $x_0 \in$ Dom f such that $f(x_0) = y_0$. Said another way, if $x_1, x_2 \in$ Dom f $(x_1 \neq x_2)$ then $f(x_1) \neq f(x_2)$. Any function f satisfying this property is said to be one to one (1–1) and has an inverse function, denoted f^{-1}:

$$x_0 \xrightarrow{f} y_0 \xrightarrow{f^{-1}} x_0$$

Definition: A function f is *one to one (1-1)* if for all $x_1, x_2 \in$ Dom f $(x_1 \neq x_2)$, then $f(x_1) \neq f(x_2)$

Not all functions are 1-1. For example, consider

$$\mathbf{f(x) = x^2}\ ;\ \mathbf{Dom\ f} = \mathbb{R}_{\mathbf{x}}\ ;\ \mathbf{Rng\ f} = [0, +\infty)_{\mathbf{y}}\ :$$



Let $\mathbf{y}_0 \in \mathbf{Rng\ f}$ $(\mathbf{y}_0 \neq 0)$**. Setting** $\mathbf{f(x) = x^2 \overset{SET}{=} y_0 \Rightarrow x \equiv \begin{cases} x_1 = -\sqrt{y_0} \\ x_2 = +\sqrt{y_0} \end{cases}}$**.**

Thus, there does NOT exist a *unique* $\mathbf{x}_0 \in \mathbf{Dom\ f}$ **such that** $\mathbf{f(x_0) = y_0}$**.**

Geometrically, a function is 1-1 if a horizontal line through every element in the range of the function intersects the graph *exactly once*. So the way we get the inverse function is to start at an element in the range (on the y-axis) of the given function, go to the graph, and then go up/down to the *unique element* on the x-axis.

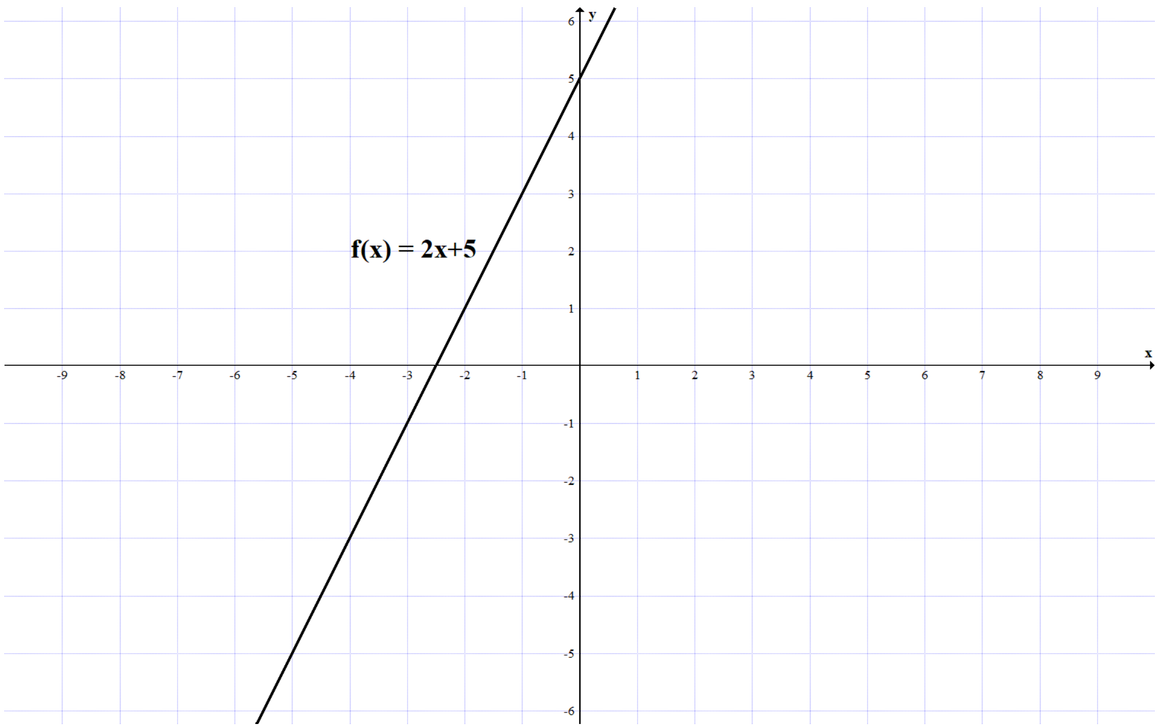
From the nature of the inverse function $\mathbf{f}^{-1}$ **, we obtain**

1. $\mathbf{Dom\ f^{-1} = Rng\ f}$
2. $\mathbf{Rng\ f^{-1} = Dom\ f}$

We also obtain the following:

If $y_0 \in \text{Dom } f^{-1}$, then $f^{-1}(y_0) = x_0$ where x_0 is the *unique element* in $\text{Dom } f$ such that $f(x_0) = y_0$.

Consider the obviously 1-1 function $y = f(x) = 2x + 5$:



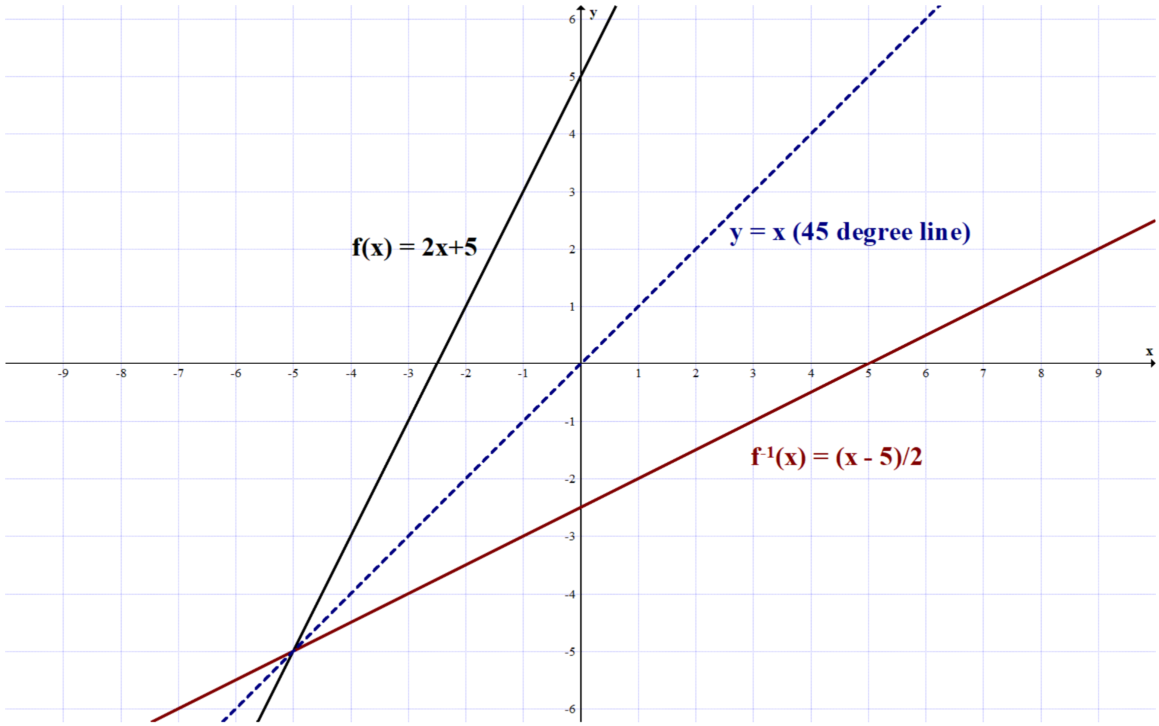
Note that this function satisfies the convention that functions are to be read starting from the x-axis, then up/down to the graph and then finally over to the y-axis, that is "correctly".

When we solve $y = f(x) = 2x + 5$ for x

$$y = 2x + 5 \Rightarrow y - 5 = 2x \Rightarrow x = \frac{y-5}{2} = f^{-1}(y)$$

we obtain a formula for $f^{-1}(y)$ read starting from the y-axis, then over to the graph, and then finally up/down to the x-axis, that is "backwards". To get the inverse function read "correctly", we just need to switch the "x" and "y":

$$f^{-1}(x) = \frac{x-5}{2}$$



The example above provides us with a two-step procedure for obtaining a formula for $y = f^{-1}(x)$ from $y = f(x)$:

Two steps:

1. **Solve $y = f(x)$ for x (Note: f^{-1} read "backwards")**
2. **Switch x and y (Note: f^{-1} read "correctly")**

There are several other items to note:

1. **$f^{-1} \neq \frac{1}{f}$. I sometimes write $f^{-1} = f^{\text{Inverse}} = f^{I}$ to avoid possible confusion.**
2. **The graph of f^{-1} read "correctly" is the graph of f (1-1) reflected in the $y = x$ line (45-degree line)**
3. **As noted earlier**
   a. **Dom f^{-1} = Rng f**
   b. **Rng f^{-1} = Dom f**
4. **Since f and f^{-1} are inverses**

$$x \xrightarrow{f} y \xrightarrow{f^{-1}} x$$

$$x \xrightarrow{f^{-1}} y \xrightarrow{f} x$$

we have

a. $f(f^{-1}(x)) = x$

b. $f^{-1}(f(x)) = x$

Using the functions above as an illustration, we have

1. $f(f^{-1}(x)) = 2f^{-1}(x) + 5 = 2\left[\dfrac{x-5}{2}\right] + 5 = x$
2. $f^{-1}(f(x)) = \dfrac{f(x)-5}{2} = \dfrac{[2x+5]-5}{2} = x$

Typical Question 01: Given $y = f(x) = \dfrac{2x}{x-4}$ **, find** f^{-1}**.**

Solution:

We have

$$\text{Dom}\, f = \mathbb{R} \setminus \{4\} = \text{Rng}\ f^{-1}$$

To find $\text{Rng}\ f$**, solve** $y = f(x)$ **for "x":**

$$y = \frac{2x}{x-4} \Rightarrow y(x-4) = 2x$$
$$\Rightarrow xy - 4y = 2x \Rightarrow xy - 2x = 4y$$
$$\Rightarrow x(y-2) = 4y \Rightarrow x = \frac{4y}{y-2} \qquad \text{(shows } f \text{ is 1-1)}$$

Thus, $\text{Rng}\, f = \mathbb{R} \setminus \{2\} = \text{Dom}\ f^{-1}$**. We use the two-step procedure to obtain** $y = f^{-1}(x)$**:**

Step 1: Solve $y = f(x)$ **for** x**: See above!**

$$x = \frac{4y}{y-2} \quad (f^{-1} \rightarrow\downarrow)$$

Step 2: Switch x **and** y

$$y = f^{-1}(x) = \frac{4x}{x-2} \qquad (f^{-1} \leftarrow\uparrow)$$

Verification #1: Analytical

1. $f^{-1}(f(x)) \stackrel{?}{=} x$

$$f^{-1}(f(x)) = \frac{4[f(x)]}{[f(x)]-2} = \frac{4\left[\frac{2x}{x-4}\right]}{\left[\frac{2x}{x-4}\right]-2} = \frac{4\left[\frac{2x}{x-4}\right]}{\left[\frac{2x}{x-4}\right]-2} * \frac{x-4}{x-4}$$

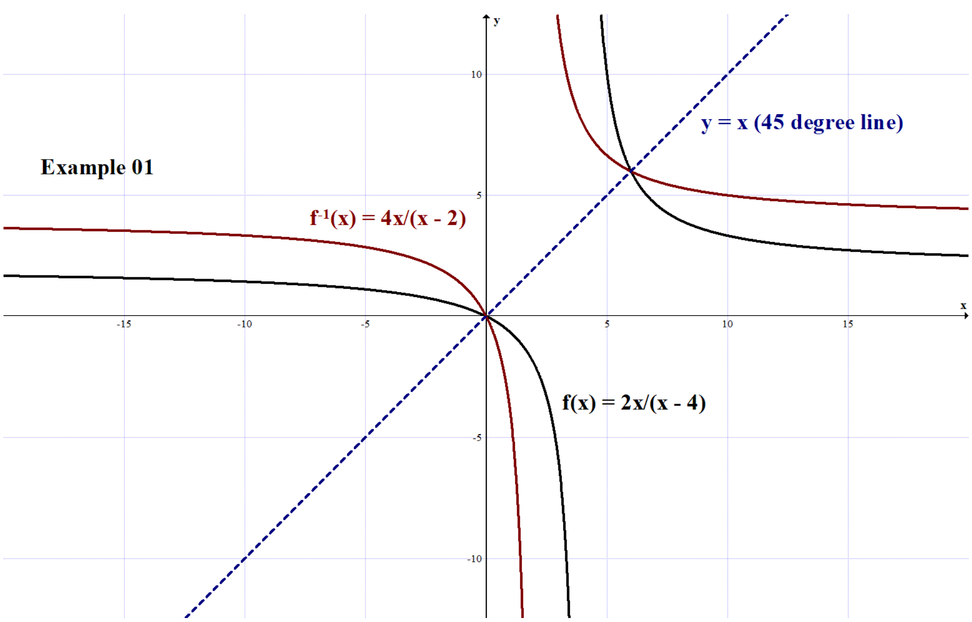
$$= \frac{8x}{2x-2(x-4)} = \frac{8x}{8} = x$$

2. $f(f^{-1}(x)) \stackrel{?}{=} x$

$$f(f^{-1}(x)) = \frac{2\left[f^{-1}(x)\right]}{\left[f^{-1}(x)\right]-4} = \frac{2\left[\frac{4x}{x-2}\right]}{\left[\frac{4x}{x-2}\right]-4} = \frac{2\left[\frac{4x}{x-2}\right]}{\left[\frac{4x}{x-2}\right]-4} * \frac{x-2}{x-2}$$

$$= \frac{8x}{4x-4(x-2)} = \frac{8x}{8} = x$$

Verification #2: Graphical



Typical Question 02: Given $y = f(x) = 2 - \sqrt[3]{x}$, find $y = f^{-1}(x)$.

Solution:

We have

$$\text{Dom} f = \mathbb{R} = \text{Rng } f^{-1}$$

To find Rng f, solve $y = f(x)$ for "x":

$$y = 2 - \sqrt[3]{x} \Rightarrow \sqrt[3]{x} = 2 - y$$

$$\Rightarrow \left(\sqrt[3]{x}\right)^3 = (2-y)^3 \Rightarrow x = (2-y)^3 \quad \text{(shows } \mathbf{f} \text{ is 1-1)}$$

Thus, $\text{Rng} f = \mathbb{R} = \text{Dom } f^{-1}$. We use the two-step procedure to obtain $y = f^{-1}(x)$:

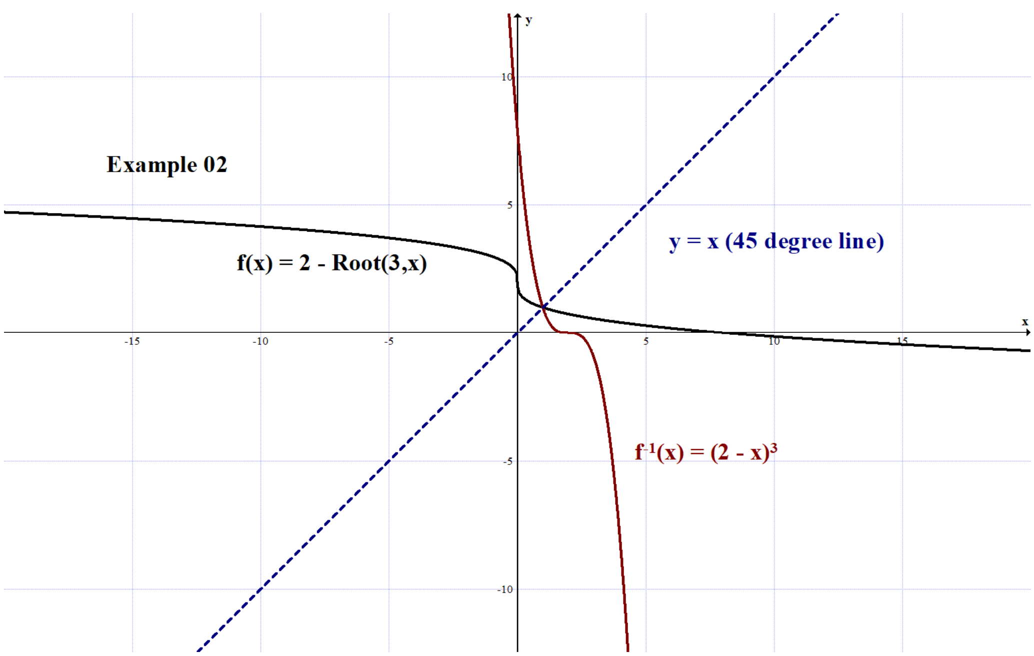
Step 1: Solve $y = f(x)$ for x: See above!

$$x = (2-y)^3 \quad (f^{-1} \rightarrow\downarrow)$$

Step 2: Switch x and y

$$y = f^{-1}(x) = (2-x)^3 \qquad (f^{-1} \leftarrow\uparrow)$$

Graphs below:



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