Determinants and Linear Equations

(n by n: "n" equations with "n" unknowns)

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1. 2x2: Linear System

$$\begin{matrix} a_{11}\ x_1 + a_{12}\ x_2 = b_1 \\ a_{21}\ x_1 + a_{22}\ x_2 = b_2 \end{matrix} \quad ; \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix};\ B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix};\ X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow AX = B$$

System Determinant: $D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{21}\,a_{12}$ (Definition)

$$D_{x_1} = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix};\ D_{x_2} = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

Cramer's Rule:

$$x_1 = \frac{D_{x_1}}{D};\ x_2 = \frac{D_{x_2}}{D} \text{ when } D \neq 0$$

Example 01:

$$l_1 : 5x - 2y = 19$$
$$l_2 : 3x + 4y = 1$$

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 5 & -2 \\ 3 & 4 \end{vmatrix} = a_{11}\,a_{22} - a_{21}\,a_{12} = (5)(4) - (3)(-2) = 26 \neq 0$$

$$D_{x_1} = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = (19)(4) - (1)(-2) = 78$$

$$D_{x_2} = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = (5)(1) - (19)(3) = -52$$

$$x_1 = \frac{D_{x_1}}{D} = \frac{78}{26} = 3$$

$$x_2 = \frac{D_{x_2}}{D} = -\frac{52}{26} = -2$$

2. 3x3: Linear System

$$\begin{aligned} a_{11}\ x_1 + a_{12}\ x_2 + a_{13}\ x_3 &= b_1 \\ a_{21}\ x_1 + a_{22}\ x_2 + a_{23}\ x_3 &= b_2 \quad \text{where} \\ a_{31}\ x_1 + a_{32}\ x_2 + a_{33}\ x_3 &= b_3 \end{aligned}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix};\ B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix};\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow AX = B$$

System Determinant: $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Using Minors: can use *any* row or column to calculate. I am choosing row 2 (Trade a 3x3 for 3 2x2's)

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{2+1} a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{2+2} a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{2+3} a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$D_{x_1} = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix};\ D_{x_2} = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix};\ D_{x_3} = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{23} & b_3 \end{vmatrix}$$

Another way to find D:

$$D_{down} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} = [a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32}]$$

$$D_{up} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} = \left[a_{31}\,a_{22}\,a_{13} + a_{32}\,a_{23}\,a_{11} + a_{33}\,a_{21}\,a_{12}\right]$$

$$D = D_{down} - D_{up}$$

Cramer's Rule:

$$x_1 = \frac{D_{x_1}}{D}\ ;\ x_2 = \frac{D_{x_2}}{D}\ ;\ x_3 = \frac{D_{x_3}}{D} \text{ when } \mathrm{D} \neq 0$$

Example 02:

$$p_1 : x - 2y + 3z = 14$$
$$p_2 : 2x + 3y - z = -7$$
$$p_3 : 4x + y + 2z = 7$$

$$D = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ 4 & 1 & 2 \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ 4 & 1 & 2 \end{vmatrix} \begin{matrix} 1 & -2 \\ 2 & 3 \\ 4 & 1 \end{matrix} \Rightarrow \left[(1)(3)(2) + (-2)(-1)(4) + (3)(2)(1)\right]$$
$$-\left[(4)(3)(3) + (1)(-1)(1) + (2)(2)(-2)\right]$$

$$D = [20] - [-27] = -7$$

$$D_x = \begin{vmatrix} 14 & -2 & 3 \\ -7 & 3 & -1 \\ 7 & 1 & 2 \end{vmatrix} \Rightarrow \begin{vmatrix} 14 & -2 & 3 \\ -7 & 3 & -1 \\ 7 & 1 & 2 \end{vmatrix} \begin{matrix} 14 & -2 \\ -7 & 3 \\ 7 & 1 \end{matrix} \Rightarrow \left[(14)(3)(2) + (-2)(-1)(7) + (3)(-7)(1)\right]$$
$$-\left[(7)(3)(3) + (1)(-1)(14) + (2)(-7)(-2)\right]$$

$$D_x = [77] - [77] = 0$$

$$D_y = \begin{vmatrix} 1 & 14 & 3 \\ 2 & -7 & -1 \\ 4 & 7 & 2 \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & 14 & 3 \\ 2 & -7 & -1 \\ 4 & 7 & 2 \end{vmatrix} \begin{matrix} 1 & 14 \\ 2 & -7 \\ 4 & 7 \end{matrix} \Rightarrow \left[(1)(-7)(2) + (14)(-1)(4) + (3)(2)(7)\right]$$
$$-\left[(4)(-7)(3) + (7)(-1)(1) + (2)(2)(14)\right]$$

$$D_y = [-28] - [-35] = 7$$

$$D_z = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ 4 & 1 & 2 \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & -2 & 14 \\ 2 & 3 & -7 \\ 4 & 1 & 7 \end{vmatrix} \begin{matrix} 1 & -2 \\ 2 & 3 \\ 4 & 1 \end{matrix} \Rightarrow \left[(1)(3)(7)+(-2)(-7)(4)+(14)(2)(1)\right]$$

$$-\left[(4)(3)(14)+(1)(-7)(1)+(7)(2)(-2)\right]$$

$$D_z = [105]-[133] = -28$$

$$x = \frac{D_x}{D} = \frac{0}{-7} = 0$$

$$y = \frac{D_y}{D} = \frac{7}{-7} = -1$$

$$z = \frac{D_z}{D} = \frac{-28}{-7} = 4$$

3. We will not solve systems of higher order using determinants …

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