# FUNctions
# Logarithmic

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**A logarithmic *function* has the form**

$\mathbf{f}(\mathbf{x}) = \log_{\mathbf{b}} \mathbf{x}$ ; $0 < \mathbf{b} < 1$ or $\mathbf{b} > 1$ **; "b" is called the base**

**Note: The restrictions on the base are because we are only considering real valued functions.**

**Note:** $\log_{\mathbf{e}} \mathbf{x} \equiv \ln \mathbf{x}$ **; natural logarithmic function**
**Base:** $\mathbf{e} \approx 2.718281828459045$

**Note:** $\log_{10} \mathbf{x} \equiv \log \mathbf{x}$ **; common logarithmic function**
**Base: 10**

**Note: Base "e"** $\left(\mathbf{y} = \log_{\mathbf{e}} \mathbf{x} = \ln \mathbf{x}\right)$ **and base "10"** $\left(\mathbf{y} = \log_{10} \mathbf{x} = \log \mathbf{x}\right)$ **can be found on most calculators. If you need to work in another base, there is a change of base formula you can use:**

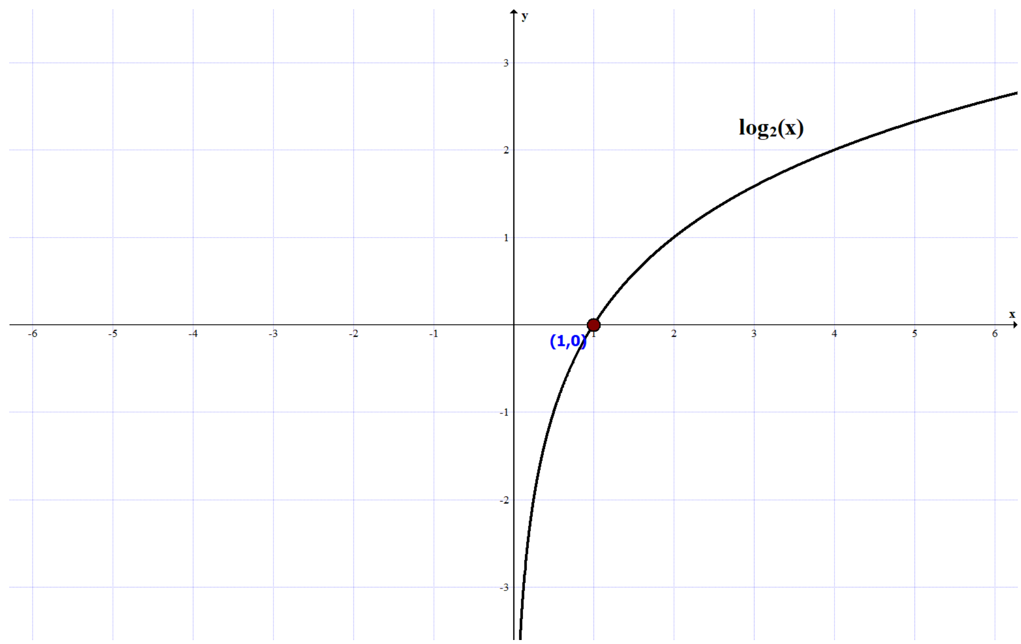
$$\mathbf{y} = \log_{\mathbf{b}} \mathbf{x} = \frac{\ln \mathbf{x}}{\ln \mathbf{b}} = \frac{\log \mathbf{x}}{\log \mathbf{b}}$$

**There are two (2) shapes a *logarithmic* function can have:**

1. $\mathbf{b} > 1$

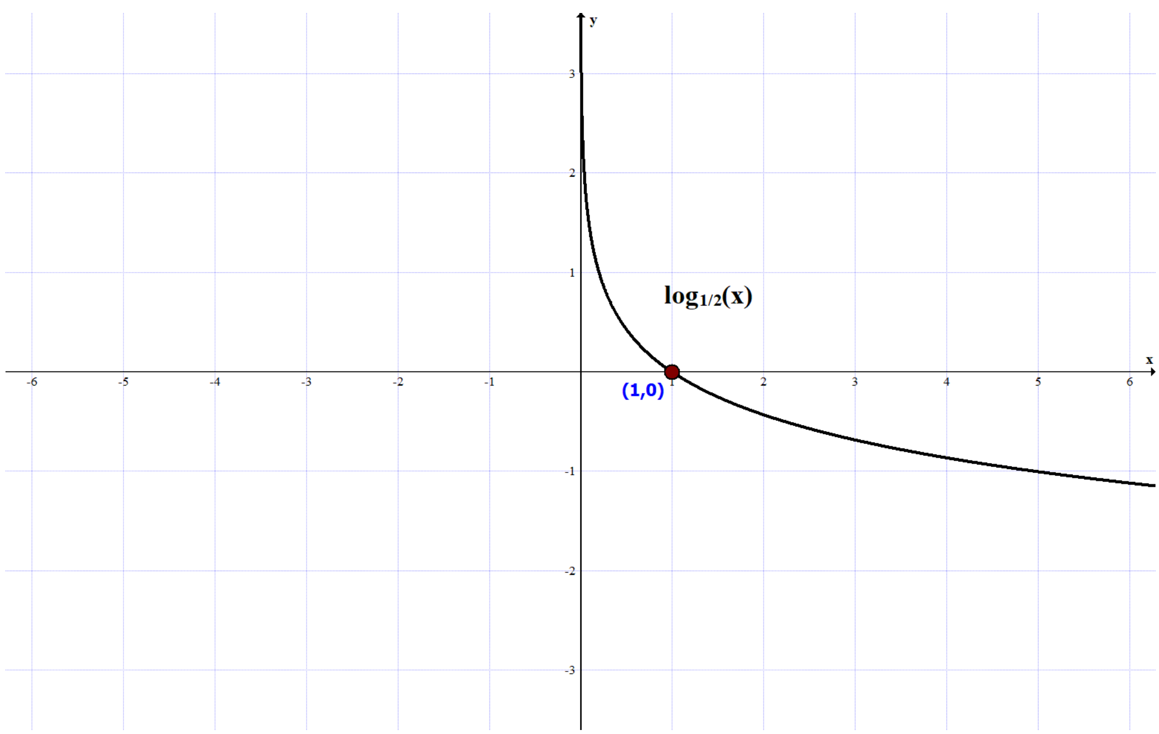
**When the base satisfies** $\mathbf{b} > 1$**, the graph has the shape of** $\mathbf{f}(\mathbf{x}) = \log_2 \mathbf{x}$**:**

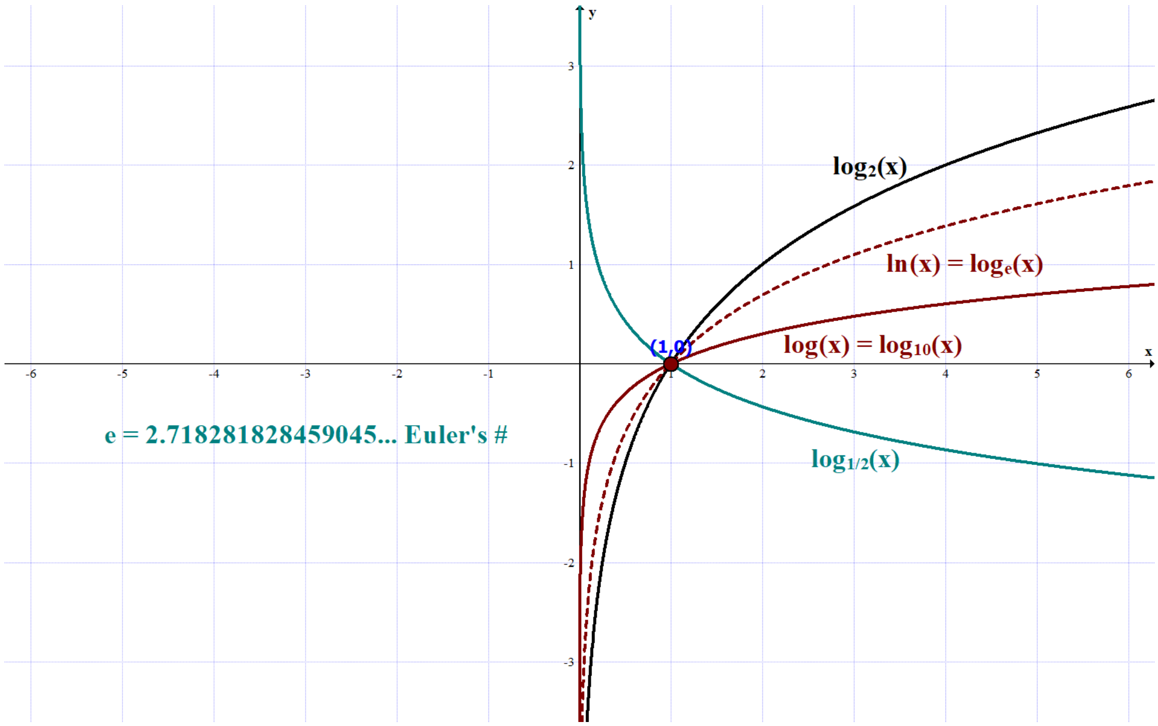



**2.** $0 < \mathbf{b} < 1$

**When the base satisfies** $0 < \mathbf{b} < 1$**, the graph has the shape of** $\mathbf{f}(\mathbf{x}) = \log_{1/2}\mathbf{x}$**:**



**Here are some graphs of the important logarithmic functions:**

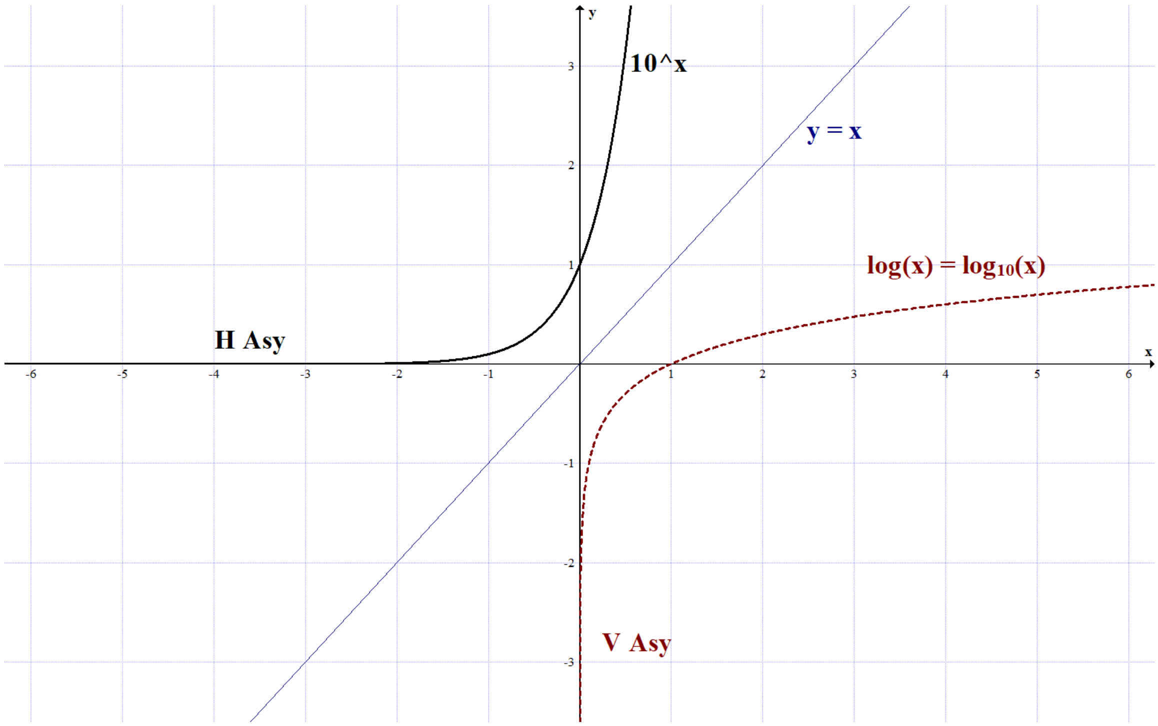


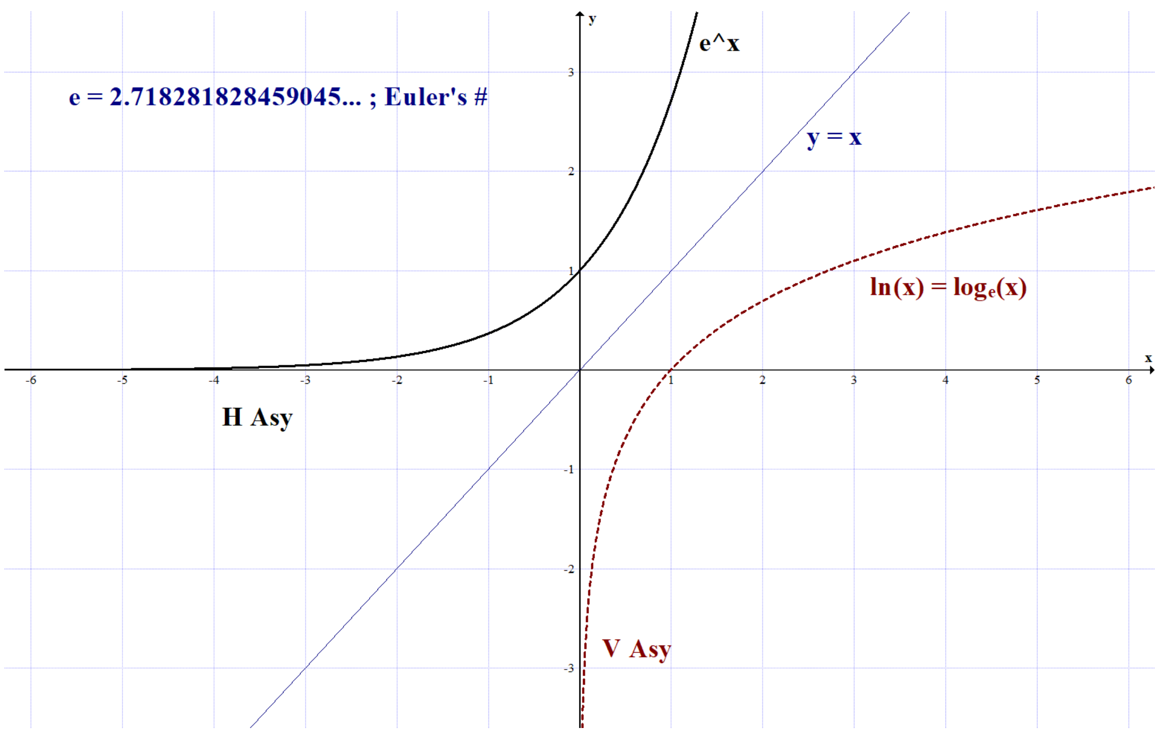

**Note:**

1. **The *domain* of every logarithmic function is $(0,+\infty)_x$. The *range* of every logarithmic function is the $(0,+\infty)_y$.**
2. **All logarithmic functions pass through $(1,0)$ since $\log_b 1 = 0$.**
3. **They all have $x = 0$ as a vertical asymptote.**

**The logarithmic functions are inverses of the 1-1 exponential functions. Among the important ones are the common log function $(y = \log_{10} x = \log x)$ and the natural log function $(y = \log_e x = \ln x)$:**


y
10^x
y = x
log(x) = log10(x)
H Asy
V Asy
x
-6
-5
-4
-3
-2
-1
1
2
3
4
5
6
3
2
1
-1
-2
-3


y
e^x
e = 2.718281828459045... ; Euler's #
y = x
ln(x) = loge(x)
H Asy
V Asy
x
-6
-5
-4
-3
-2
-1
1
2
3
4
5
6
3
2
1
-1
-2
-3

**As the table repeated from the exponential notes shows, for each exponential property, there is a corresponding logarithmic property:**

| | Exponential | Logarithmic |
|---|---|---|
| | **(One Base)** | |
| **1** | $b^0 = 1$ | $\log_b 1 = 0$ |
| **2** | $b^1 = b$ | $\log_b b = 1$ |
| **3** | $b^x = b^y \Leftrightarrow x = y$ **(1-1)** | $\log_b x = \log_b y \Leftrightarrow x = y$ **(1-1)** |
| **4** | $\log_b b^x = x$ **(Inverse)** | $b^{\log_b x} = x$ **(Inverse)** |
| **5** | $b^x = y \Leftrightarrow x = \log_b y$<br>**Trade exponential for logarithmic** | $x = \log_b y \Leftrightarrow b^x = y$<br>**Trade logarithmic for exponential** |
| **6** | $\left(b^x\right)^y = b^{x*y}$ | $\log_b x^y = y * \log_b x$<br>**Trade exponential for multiplication** |
| **7**$\neq$ | $\left(b^x\right)^y \neq y * b^x$ | $\left(\log_b x\right)^y \neq y * \log_b x$ |
| **8** | $b^x * b^y = b^{x+y}$ | $\log_b\left(x * y\right) = \log_b x + \log_b y$<br>**Trade multiplication for addition** |
| **9**$\neq$ | $b^{x+y} \neq b^x + b^y$ | $\log_b\left(x + y\right) \neq \log_b x + \log_b y$ |
| **10**$\neq$ | $b^{x-y} \neq b^x - b^y$ | $\log_b\left(x - y\right) \neq \log_b x - \log_b y$ |
| **11** | $\frac{b^x}{b^y} = b^{x-y} = \frac{1}{b^{y-x}}$ | $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$<br>**Trade division for subtraction** |
| **12**$\neq$ | $\frac{b^x}{b^y} \neq \frac{x}{y}$ | $\frac{\log_b x}{\log_b y} \neq \frac{x}{y}$ |
| **13** | | $\log_b x = \frac{\log_{10} x}{\log_{10} b} = \frac{\log_e x}{\log_e b}$<br>**Change of Base** |
| | **(Two Bases)** | |
| **14** | $\left(a * b\right)^x = a^x * b^x$ | |

| | | |
|---|---|---|
| **15** | $\left(\frac{\mathbf{a}}{\mathbf{b}}\right)^{\mathbf{x}} = \frac{\mathbf{a}^{\mathbf{x}}}{\mathbf{b}^{\mathbf{x}}}$ | |
| **16** | $\mathbf{b}^{-\mathbf{x}} = \frac{1}{\mathbf{b}^{\mathbf{x}}}\ ;\ \frac{1}{\mathbf{b}^{-\mathbf{x}}} = \mathbf{b}^{\mathbf{x}}$ | |
| **17** | $\left(\frac{\mathbf{a}}{\mathbf{b}}\right)^{-\mathbf{x}} = \left(\frac{\mathbf{b}}{\mathbf{a}}\right)^{\mathbf{x}}$ | |
| **18** $\neq$ | $(\mathbf{a}+\mathbf{b})^{\mathbf{x}} \neq \mathbf{a}^{\mathbf{x}} + \mathbf{b}^{\mathbf{x}}$ | |
| **19** $\neq$ | $(\mathbf{a}-\mathbf{b})^{\mathbf{x}} \neq \mathbf{a}^{\mathbf{x}} - \mathbf{b}^{\mathbf{x}}$ | |
| **20** | $\mathbf{a}^{\mathbf{x}}\mathbf{b}^{\mathbf{y}} = \mathbf{a}^{\mathbf{x}}\mathbf{b}^{\mathbf{y}}$ | |

**We now use these properties to solve logarithmic and selected exponential equations:**

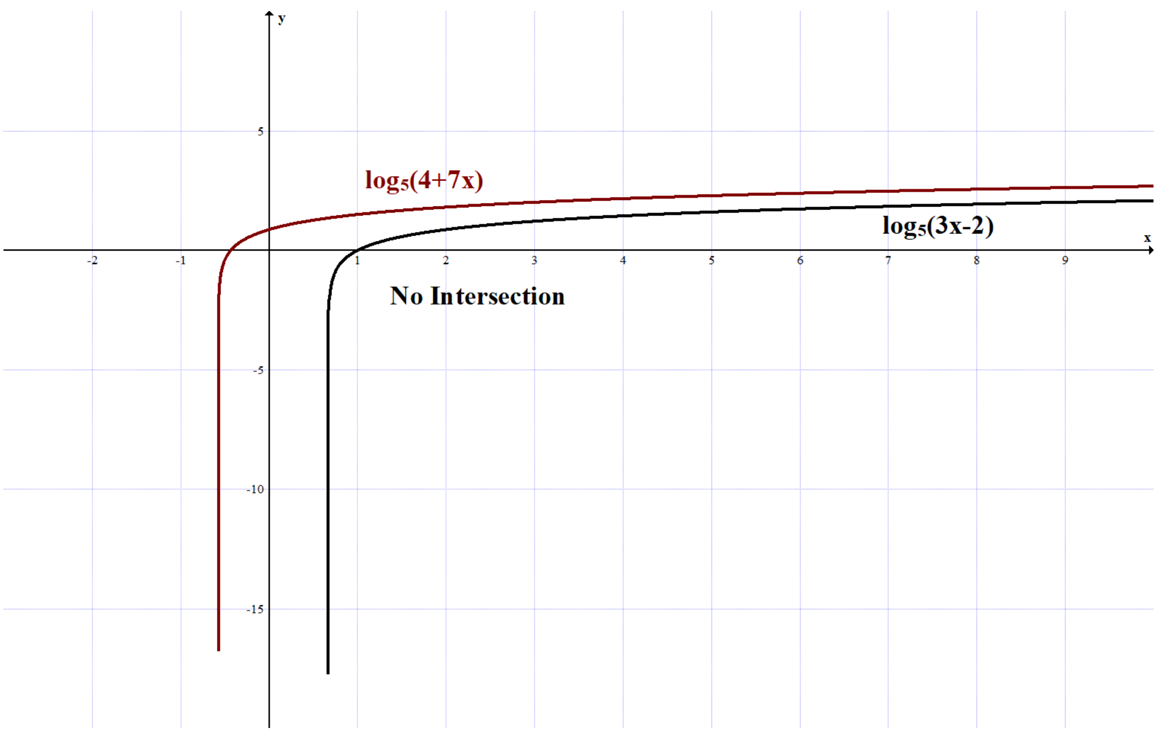
**Typical Question 01:**

$\log_5(3\mathbf{x}-2)=\log_5(4+7\mathbf{x}) \Rightarrow \mathbf{x}=?$

Property Log 03 $\Rightarrow 3\mathbf{x}-2=4+7\mathbf{x} \Rightarrow -6=4\mathbf{x} \Rightarrow \mathbf{x}=-\frac{3}{2}$

Although the linear equation has a solution, since logs do NOT allow negatives ($\log_5\left(-\frac{13}{2}\right)$ is NOT a real number) , there is no solution to the original equation.

**Visual Solution:**




**Note:** $\log_5(3(-3/2)-2)=\log_5\left(-\frac{13}{11}\right)\notin\mathbb{R}$

## Typical Question 02: (From Exponential Notes)

$5^{3x-4} = 17 \Rightarrow x = ?$ (Solution is UGLY!)

Property Log 03 $\Rightarrow \ln 5^{3x-4} = \ln 17$

Property Log 06 $\Rightarrow (3x-4) * \ln 5 = \ln 17$

$$\Rightarrow 3x - 4 = \frac{\ln 17}{\ln 5} \Rightarrow 3x = \frac{\ln 17}{\ln 5} + 4 \Rightarrow x = \frac{\ln 17}{3\ln 5} + \frac{4}{3} \approx 1.92012$$

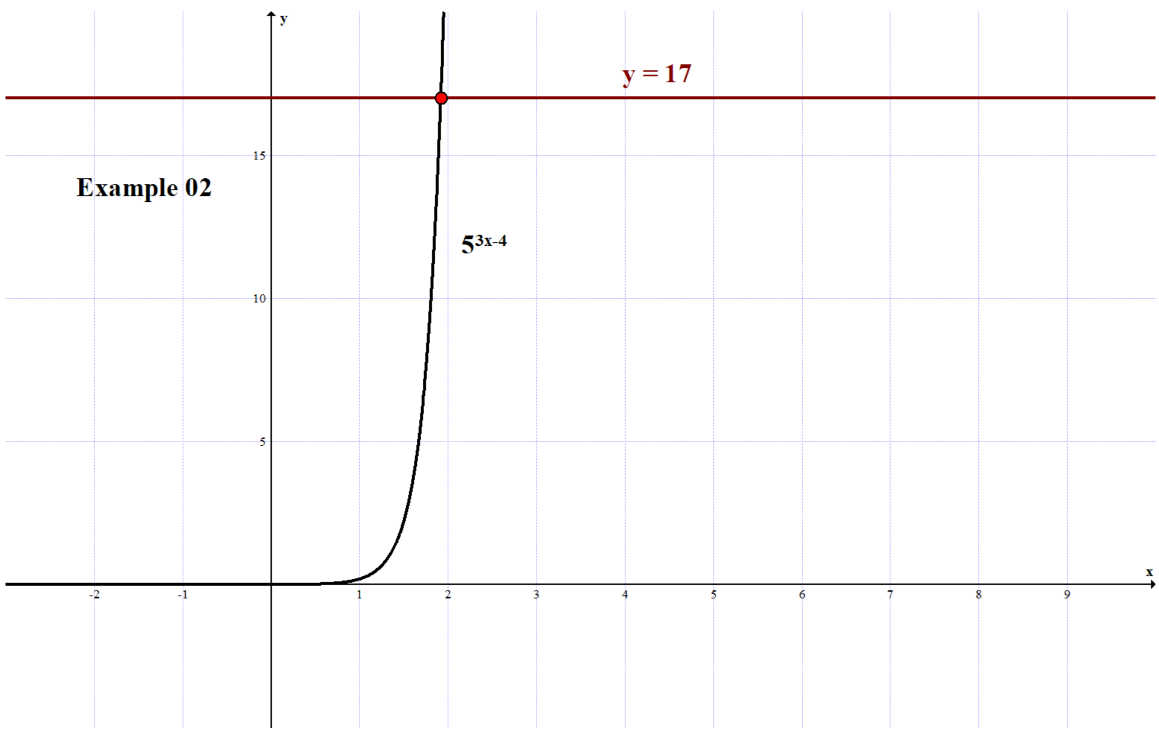
We can further simplify the answer using other logarithmic properties:

Property Log 06 $\Rightarrow x = \dfrac{\frac{\ln 17}{\ln 5} + 4}{3} = \dfrac{\ln 17 + 4\ln 5}{3\ln 5} = \dfrac{\ln 17 + \ln 5^4}{\ln 5^3} = \dfrac{\ln 17 + \ln 625}{\ln 125}$

Property Log 08 $\Rightarrow x = \dfrac{\ln 10625}{\ln 125} \approx 1.92012$

Check: $5^{3(1.92012)-4} \cong 17$

## Visual Solution:



## Typical Question 03:

$5^x = 17 \Rightarrow x = ?$

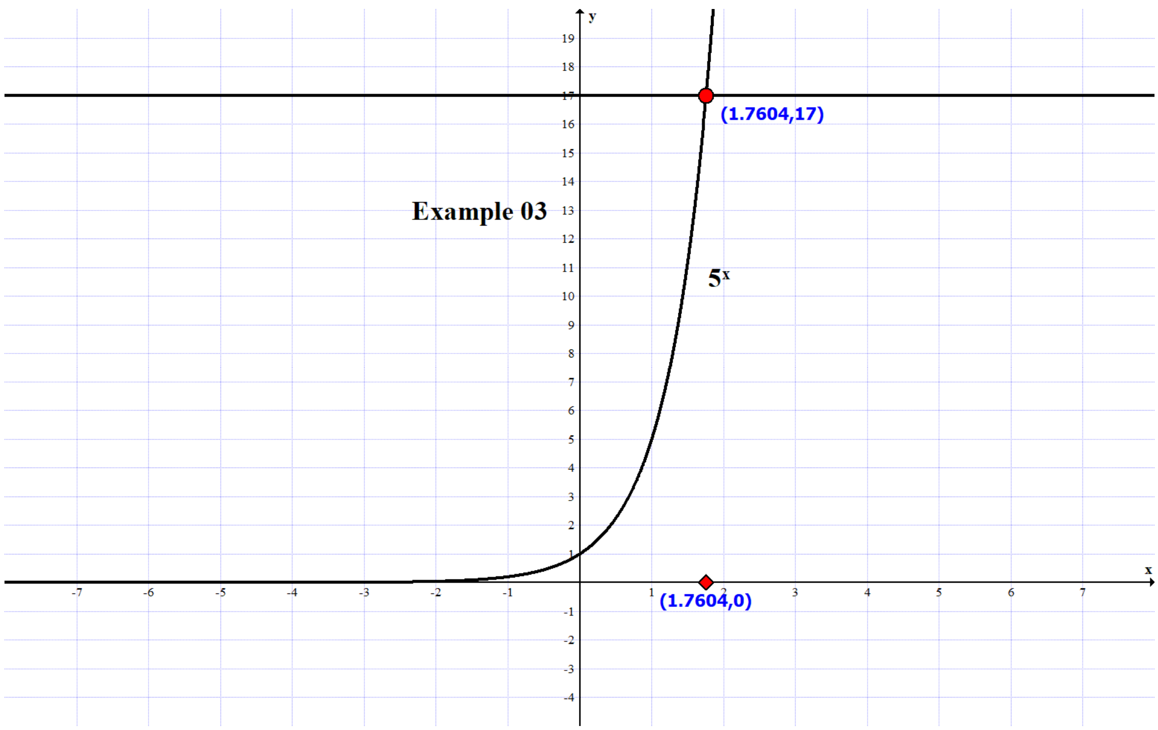
Property Log 03 $\Rightarrow \ln 5^x = \ln 17$

Property Log 06 $\Rightarrow x * \ln 5 = \ln 17$

$\Rightarrow x = \dfrac{\ln 17}{\ln 5} \Rightarrow x = 1.7604$

Check: $5^{1.7604} \cong 17$

## Visual Solution:



**Typical Question 04:**

$3^{2-3\mathbf{x}} = 7^{2\mathbf{x}-11} \Rightarrow \mathbf{x} = ?$ (Solution is SUPER UGLY!)

Property Log 03 $\Rightarrow \ln 3^{2-3\mathbf{x}} = \ln 7^{2\mathbf{x}-11}$

Property Log 06 $\Rightarrow (2-3\mathbf{x}) * \ln 3 = (2\mathbf{x}-11) * \ln 7$

$\Rightarrow 2\ln 3 - 3\mathbf{x}\ln 3 = 2\mathbf{x}\ln 7 - 11\ln 7$ (Ugly linear equation!)

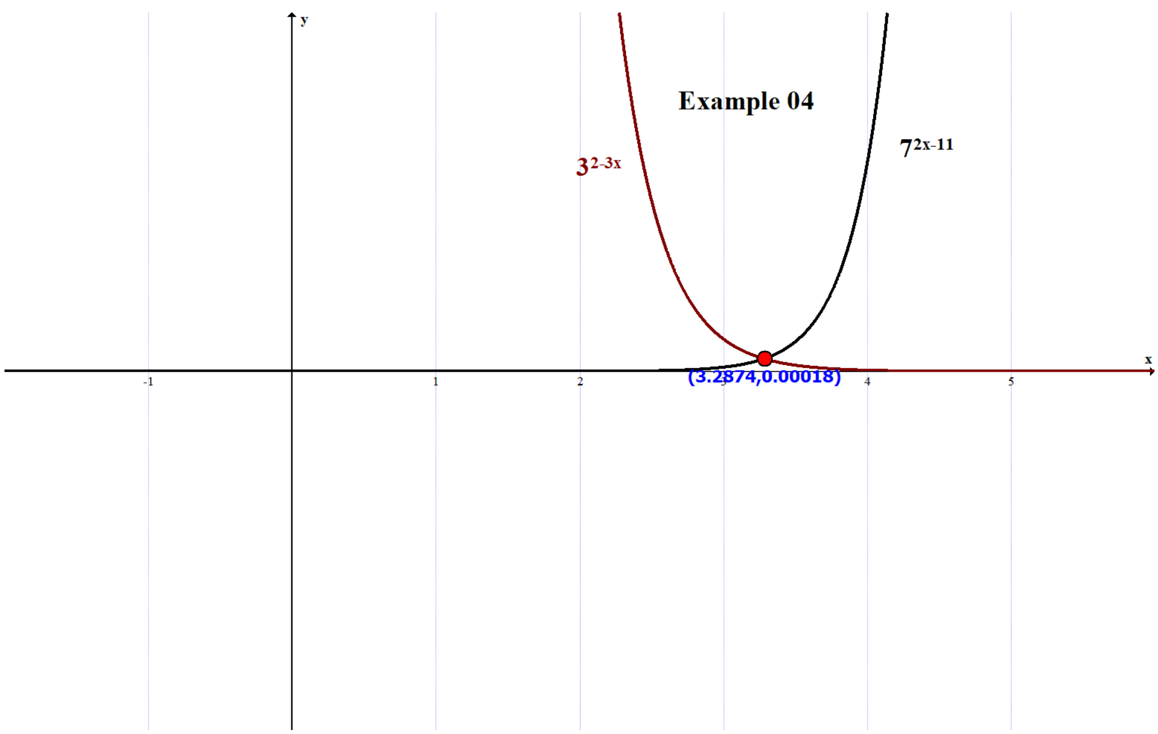
$\Rightarrow 2\ln 3 + 11\ln 7 = 3\mathbf{x}\ln 3 + 2\mathbf{x}\ln 7$

$\Rightarrow 2\ln 3 + 11\ln 7 = (3\ln 3 + 2\ln 7)\mathbf{x}$

Property Log 05 & 08 $\Rightarrow \mathbf{x} = \dfrac{2\ln 3 + 11\ln 7}{3\ln 3 + 2\ln 7} = \dfrac{\ln(3^2 * 7^{11})}{\ln(3^3 * 7^2)}$

Property Log 08 $\Rightarrow \mathbf{x} = \dfrac{\ln 17795940687}{\ln 1323} \approx 3.28372$

**Visual Solution:**



**Typical Question 05:**

$2^{1-2\mathbf{x}} = 10^{\mathbf{x}} \Rightarrow \mathbf{x} = ?$

Property Log 03 $\Rightarrow \log_{10} 2^{1-2\mathbf{x}} = \log_{10} 10^{\mathbf{x}}$

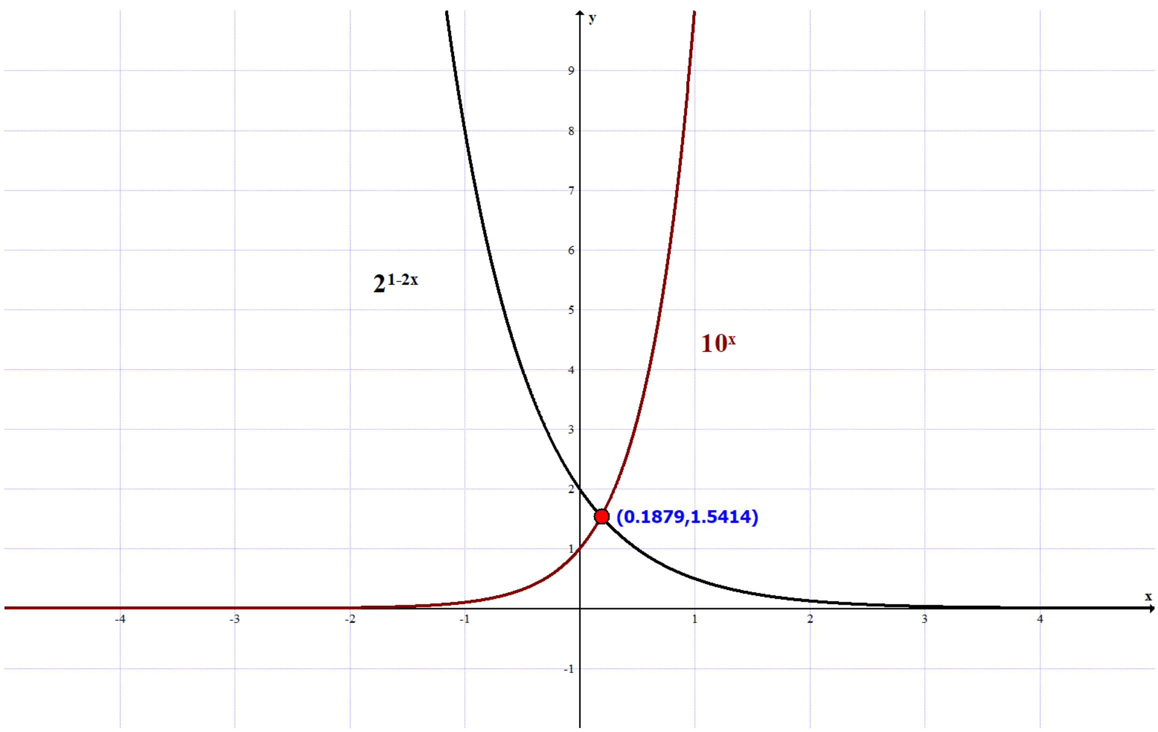
Property Log 06 $\Rightarrow (1-2\mathbf{x}) * \log_{10} 2 = \mathbf{x} * \log_{10} 10 = \mathbf{x}$ ; Property Log 02

$\Rightarrow \log_{10} 2 - (2\log_{10} 2)\mathbf{x} = \mathbf{x} \Rightarrow \mathbf{x}(1 + 2\log_{10} 2) = \log_{10} 2$

$$\Rightarrow \mathbf{x} = \frac{\log_{10} 2}{1 + 2\log_{10} 2} \approx 0.1879$$

Check: $2^{1-2(0.1879)} \approx 1.5414 \approx 10^{0.1879}$

**Visual Solution:**



**Typical Question 06:**

$\log_3(2\mathbf{x}-5)=4 \Rightarrow \mathbf{x}=?$

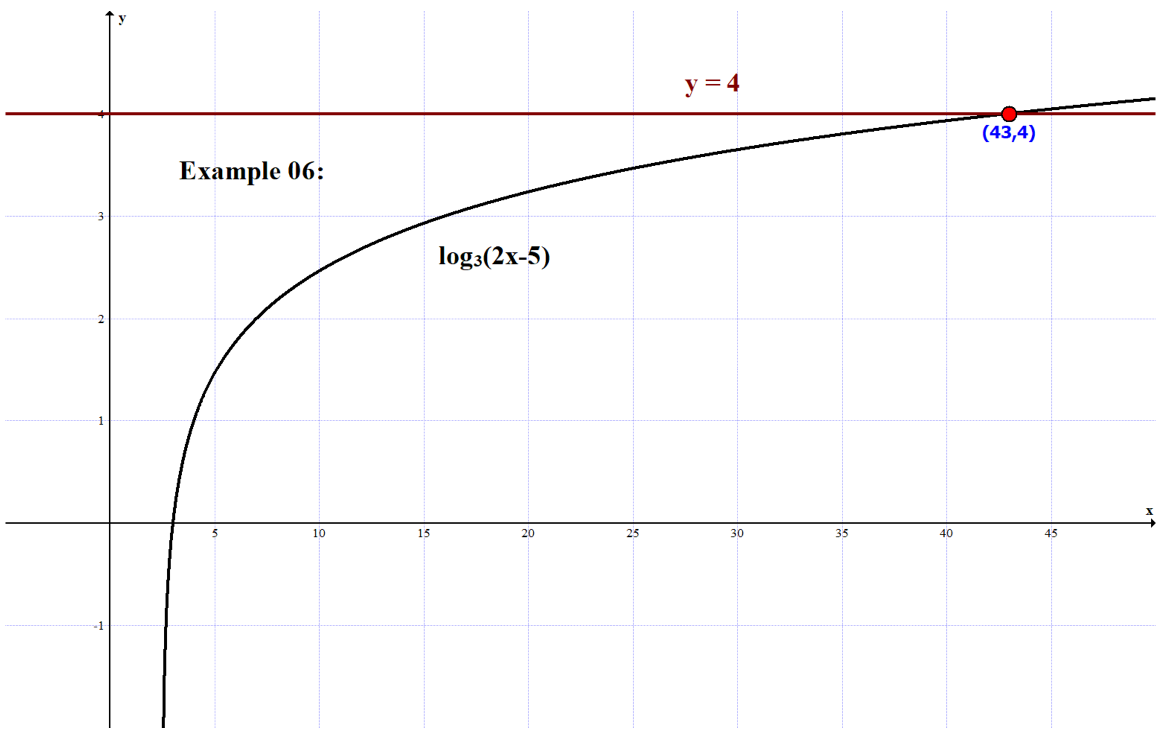
Property Exp 03 $\Rightarrow 3^{\log_3(2\mathbf{x}-5)}=3^4$

Property Log 04 $\Rightarrow 2\mathbf{x}-5=3^4=81$

$\Rightarrow 2\mathbf{x}=81+5=86 \Rightarrow \mathbf{x}=43$

Check: $\log_3(2*43-5)=\log_3 81=\log_3 3^4=4*\log_3 3=4$

**Visual Solution:**



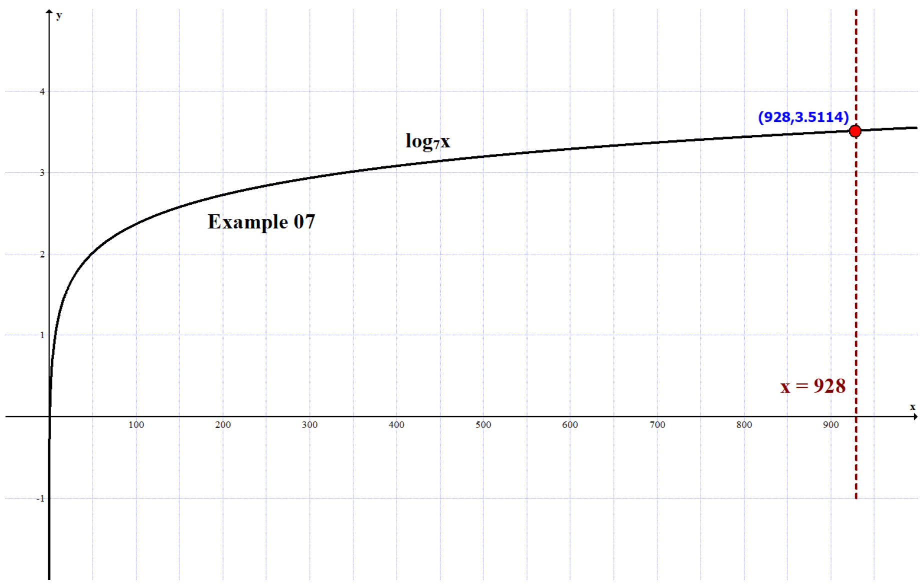
**Typical Question 07:**

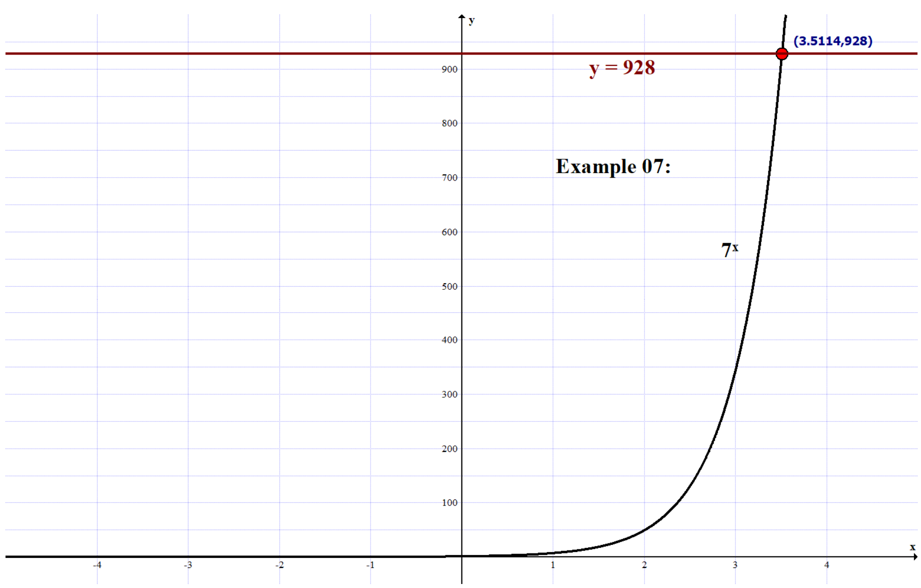
$\log_7 928 = \mathbf{x} = ?$

$$\text{Property Log 13} \Rightarrow \mathbf{x} = \log_7 928 = \frac{\ln 928}{\ln 7} \approx 3.5114$$

Check: $7^{3.5114} \approx 928$ ; Property 05

**Visual Solution:**



**Typical Question 08:**

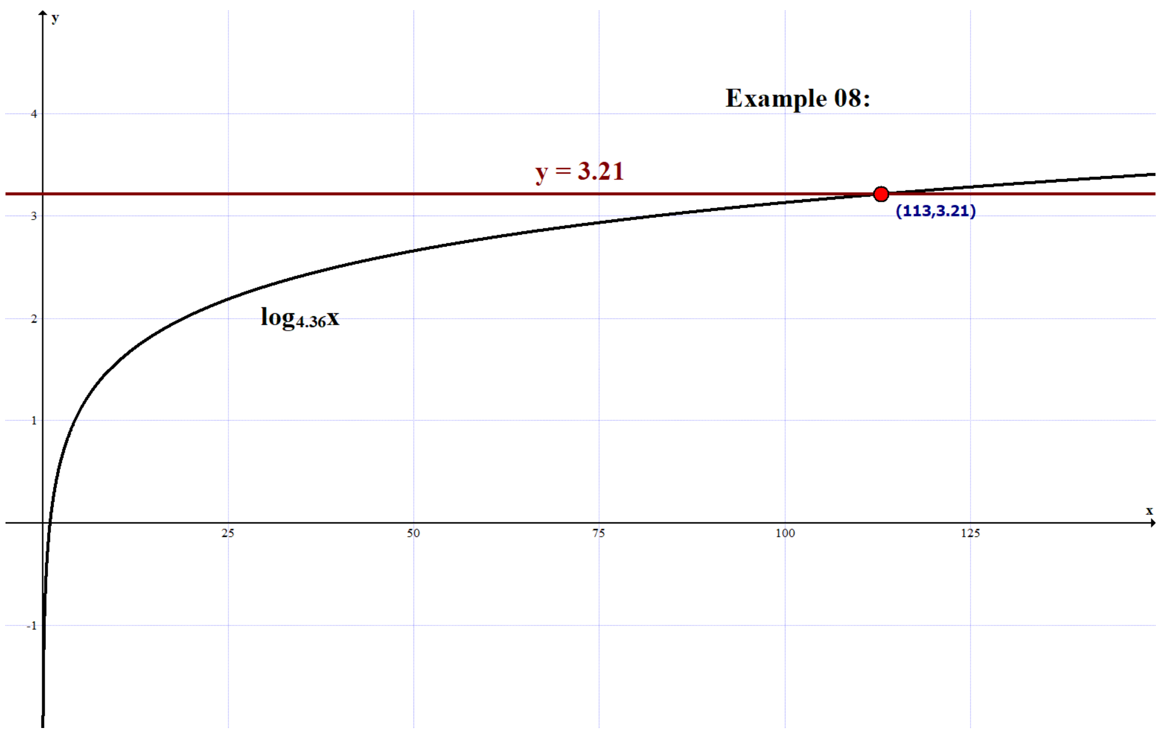
$\log_{x} 113 = 3.21 \Rightarrow \mathbf{x} = ?$

Property Log 04 $\Rightarrow 113 = \mathbf{x}^{\log_{x} 113} = \mathbf{x}^{3.21}$

$\Rightarrow (113)^{\frac{1}{3.21}} = \left(\mathbf{x}^{3.21}\right)^{\frac{1}{3.21}} = \mathbf{x}$

$\Rightarrow \mathbf{x} \approx (113)^{0.3115} \approx 4.36047$

**Visual Solution:**



**Typical Question 09:**

$$\log_2(2\mathbf{x}-5)-\log_2(3-7\mathbf{x})=3 \Rightarrow \mathbf{x}=?$$

$$\left[\text{Also: } \log_2(2\mathbf{x}-5)=3+\log_2(3-7\mathbf{x})\right]$$

WRONG: Can NOT use Property 03 - TWO terms!

$$\left(\begin{array}{l} \not\Rightarrow 2^{\log_2(2\mathbf{x}-5)}-2^{\log_2(3-7\mathbf{x})}=2^3 \\ (2\mathbf{x}-5)-(3-7\mathbf{x})=8 \\ 9\mathbf{x}-8=8 \\ 9\mathbf{x}=16 \\ \mathbf{x}=\frac{16}{9} \quad \text{TRASH!} \end{array}\right)$$

RIGHT: Now we can use Property 03 - ONLY One Term!

$$\log_2(2\mathbf{x}-5)-\log_2(3-7\mathbf{x})=3$$

$$\text{Property Log 11} \Rightarrow \log_2\left(\frac{2\mathbf{x}-5}{3-7\mathbf{x}}\right)=3$$

$$\text{Property Exp 03} \Rightarrow 2^{\log_2\left(\frac{2\mathbf{x}-5}{3-7\mathbf{x}}\right)}=2^3$$

$$\text{Property Log 04} \Rightarrow \frac{2\mathbf{x}-5}{3-7\mathbf{x}}=8$$

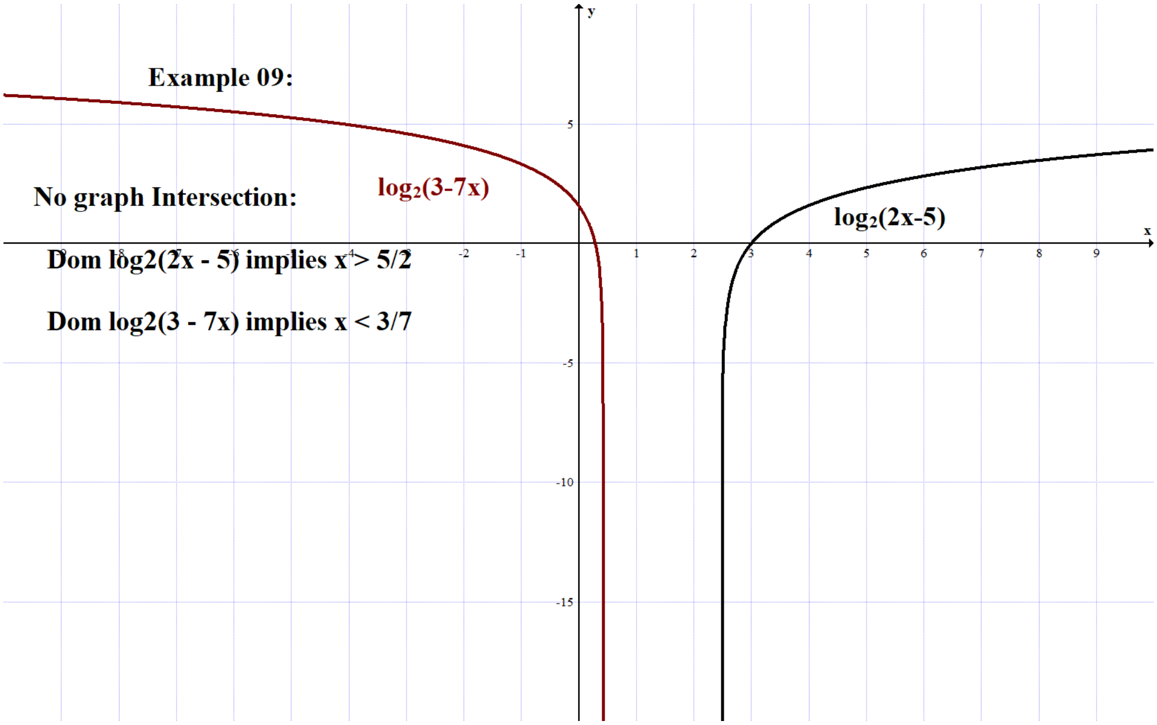
$$\Rightarrow 2\mathbf{x}-5=24-56\mathbf{x} \Rightarrow 58\mathbf{x}=29$$

$$\Rightarrow \mathbf{x}=\frac{29}{58}=\frac{1}{2} \Rightarrow \text{No Solution}$$

$$\text{since } \log_2\left(2*\left(\frac{1}{2}\right)-5\right)=\log_2(-4) \text{ is NOT a real number}$$

**Visual Solution:** $\textbf{Dom}\left[\log_2\left(2\mathbf{x}-5\right)-\log_2\left(3-7\mathbf{x}\right)\right]=\phi$



**Typical Question 10:**

$$\log_2(2\mathbf{x}+3)-\log_2(3+5\mathbf{x})=4\Rightarrow \mathbf{x}=?$$

$$\text{Property Log 11}\Rightarrow \log_2\left(\frac{2\mathbf{x}+3}{3+5\mathbf{x}}\right)=4$$

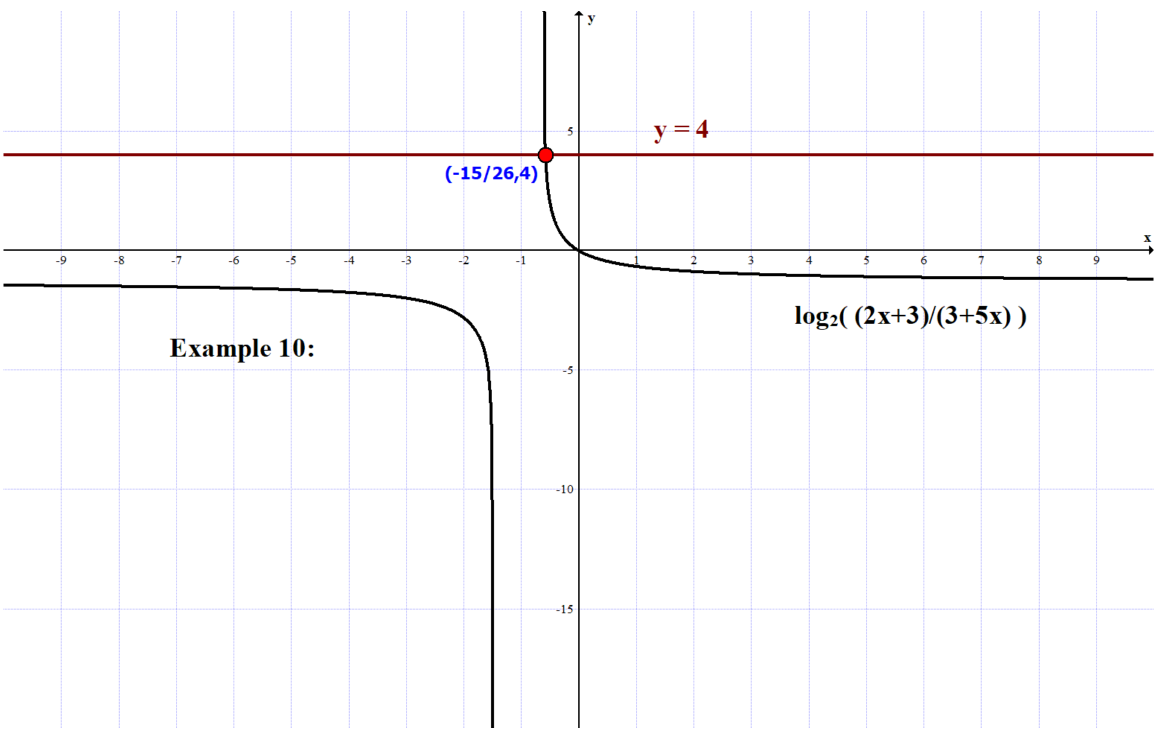
$$\text{Property Exp 03}\Rightarrow 2^{\log_2\left(\frac{2\mathbf{x}+3}{3+5\mathbf{x}}\right)}=2^4$$

$$\text{Property Log 04}\Rightarrow \frac{2\mathbf{x}+3}{3+5\mathbf{x}}=16\Rightarrow 2\mathbf{x}+3=16(3\mathbf{x}+5)$$

$$\Rightarrow 2\mathbf{x}+3=48+80\mathbf{x}\Rightarrow -45=78\mathbf{x}$$

$$\Rightarrow \mathbf{x}=-\frac{45}{78}=-\frac{15}{26}$$

**Visual Solution:**



**Typical Question 11:**

$\log_3(2\mathbf{x}+3)+\log_3(2\mathbf{x}-3)=2 \Rightarrow \mathbf{x}=?$

Property Log 08 $\Rightarrow \log_3\left[(2\mathbf{x}+3)(2\mathbf{x}-3)\right]=2$

Property Exp 03 $\Rightarrow 3^{\log_3\left[(2\mathbf{x}+3)(2\mathbf{x}-3)\right]}=3^2$

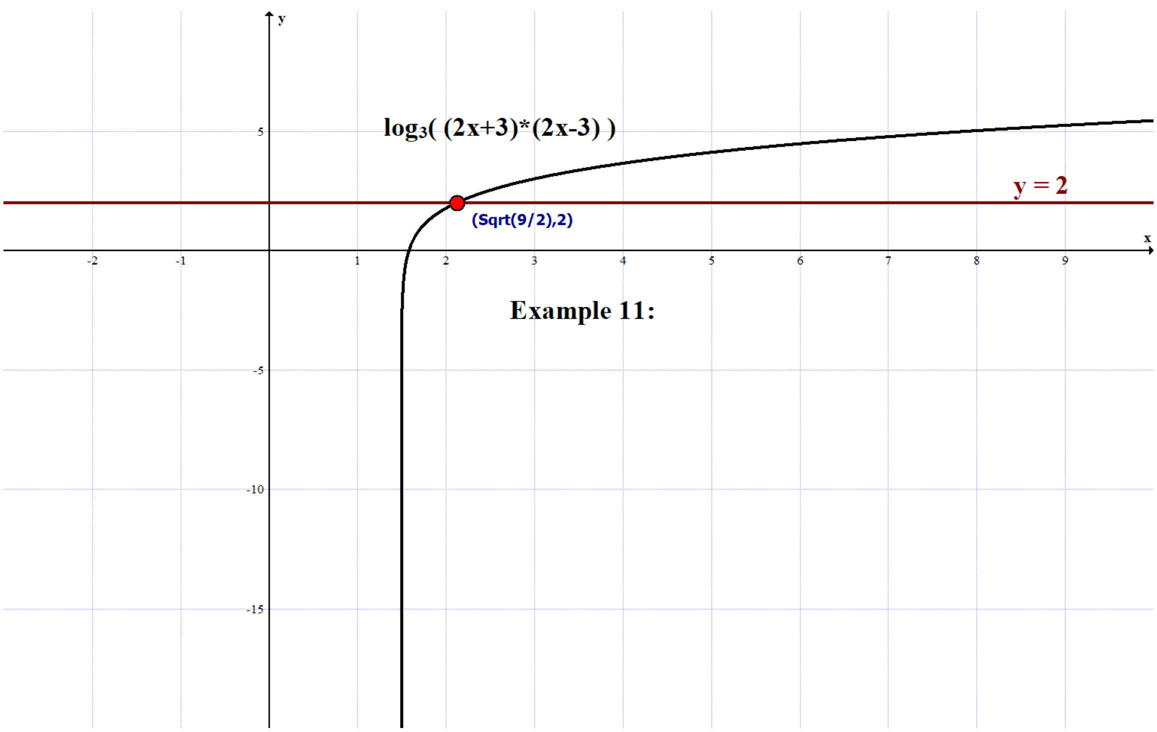
Property Log 04 $\Rightarrow (2\mathbf{x}+3)(2\mathbf{x}-3)=9$

$\Rightarrow 4\mathbf{x}^2-9=9 \Rightarrow 4\mathbf{x}^2=18 \Rightarrow \mathbf{x}^2=\frac{9}{2}$

$\Rightarrow \mathbf{x}=\pm\sqrt{\frac{9}{2}} \approx \pm 2.1213 \Rightarrow \mathbf{x}=+\sqrt{\frac{9}{2}} \approx +2.1213$

since $-\sqrt{\frac{9}{2}}$ is NOT in the domain of the function.

**Visual Solution:**



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