Matrices
Systems of Linear Equations
(n by m: "n" equations with "m" unknowns)

$$\begin{bmatrix} \text{MATH by Wilson} \\ \text{Your Personal Mathematics Trainer} \\ \text{MathByWilson.com} \end{bmatrix}$$

We have used Equivalent *Equation* Operations

1. Interchange two equations: $Eq_i \leftrightarrow Eq_j$
2. Multiply an equation by $k \neq 0$: $k * Eq_i \to Eq_i$
3. Multiply an equation by $k \neq 0$ and add the result to another equation: $k * Eq_i + Eq_j \to Eq_j$

and the corresponding Equivalent *Row* Operations

1. Interchange two rows: $R_i \leftrightarrow R_j$
2. Multiply a row by $k \neq 0$: $k * R_i \to R_i$
3. Multiply a row by $k \neq 0$ and add the result to another row: $k * R_i + R_j \to R_j$

to solve "nice" linear systems of equations or its Augmented Matrix form:

$\begin{aligned} 2x - 3y &= 1 \\ x + 4y &= 6 \end{aligned}$	$\left[\begin{array}{cc\|c} 2 & -3 & 1 \\ 1 & 4 & 6 \end{array}\right]$

Notice, because this system is "nice" – one solution – we were able to transform the "System Coefficients" portion of the Augmented Matrix $\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$ into its Reduced Row Echelon Form (also called the Gauss-Jordan Form) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

$2x-3y=1$ $x+4y=6$	$\left[\begin{array}{cc\|c} 2 & -3 & 1 \\ 1 & 4 & 6 \end{array}\right]$
$Eq_2 \leftrightarrow Eq_1$	$R_2 \leftrightarrow R_1$
$x+4y=6$ $2x-3y=1$	$\left[\begin{array}{cc\|c} 1 & 4 & 6 \\ 2 & -3 & 1 \end{array}\right]$
$-2Eq_1 + Eq_2 \rightarrow Eq_2$	$-2R_1 + R_2 \rightarrow R_2$
$x+\ 4y=\ \ 6$ $0x-11y=-11$	$\left[\begin{array}{cc\|c} 1 & 4 & 6 \\ 0 & -11 & -11 \end{array}\right]$
$-\frac{1}{11}Eq_2 \rightarrow Eq_2$	$-\frac{1}{11}R_2 \rightarrow R_2$
$x+4y=6$ $0x+\ y=1$	$\left[\begin{array}{cc\|c} 1 & 4 & 6 \\ 0 & 1 & 1 \end{array}\right]$
$-4Eq_2 + Eq_1 \rightarrow Eq_1$	$-4R_2 + R_1 \rightarrow R_1$
$x+0y=2$ $0x+\ y=1$	$\left[\begin{array}{cc\|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$

In general, the **Reduced Row-Echelon Form** of a matrix has

1. Its rows of zeroes, if any, are at the bottom of the matrix.
2. The first element of a non-zero row is a “1”, called the “first 1” in that row
3. The first one in a non-zero row is to the right of the first 1 in any non-zero row above it.
4. The elements above/below the first 1 in each non-zero row are zero.

Note: The n by n system will have a unique solution when the System Coefficients portion of the Augmented Matrix can be transformed into the Identity Matrix. If the system n by m ($n \neq m$**)** or we do not obtain the Identity Matrix, we can determine the solutions, if any, using the Row-Echelon Form of the System Coefficients portion of the Augmented Matrix. In general, the **Row-Echelon Form** satisfies

1. Its rows of zeroes, if any, are at the bottom of the matrix.
2. The first element of a non-zero row is a “1”, called the “first 1” in that row

3. The first one in a non-zero row is to the right of the first 1 in any non-zero row above it.
4. Condition 4 above is omitted.

Example 01: Solve
$$\begin{aligned} 3x - y + z &= 9 \\ x + 2y - 3z &= -5 \\ 4x + y - 2z &= 4 \end{aligned}$$

Solution:
We have

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 9 \\ 1 & 2 & -3 & -5 \\ 4 & 1 & -2 & 4 \end{array}\right]$$

$$R_2 \leftrightarrow R_1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & -5 \\ 3 & -1 & 1 & 9 \\ 4 & 1 & -2 & 4 \end{array}\right]$$

$$\left.\begin{array}{l} -3R_1 + R_2 \to R_2 \\ -4R_1 + R_3 \to R_3 \end{array}\right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & -5 \\ 0 & -7 & 10 & 24 \\ 0 & -7 & 10 & 24 \end{array}\right]$$

$$-R_2 + R_3 \to R_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & -5 \\ 0 & -7 & 10 & 24 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$-\frac{1}{7}R_2 \to R_2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & -5 \\ 0 & 1 & -\frac{10}{7} & -\frac{24}{7} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since we are not going to get a unique solution, we will use the Row-Echelon Form to rewrite the system's xyz form:

$$\begin{aligned} x + 2y - 3z &= -5 \\ y - \frac{10}{7}z &= -\frac{24}{7} \end{aligned}$$

Thus

$$y = -\frac{24}{7} + \frac{10}{7}z$$

&

$$x = -2y + 3z - 5 = -2\left(-\frac{24}{7} + \frac{10}{7}z\right) + 3z - 5$$

$$= \frac{13}{7} + \frac{1}{7}z$$

Setting $t = z$, ("t" is called the parameter), we have

$$x = \frac{13}{7} + \frac{1}{7}t$$

$$y = -\frac{24}{7} + \frac{10}{7}t$$

$$z = t$$

OR

$$\left(\frac{13}{7} + \frac{1}{7}t, -\frac{24}{7} + \frac{10}{7}t, t\right)$$

and we obtain an infinite number of solutions, on for each $t \in \mathbb{R}$. A few solutions are

1. $t = -1: \left(\frac{12}{7}, -\frac{34}{7}, -1\right)$
2. $t = 0: \left(\frac{13}{7}, -\frac{24}{7}, 0\right)$
3. $t = 1: (2, -2, 1)$

Example 02: Solve
$$\begin{aligned} 3x - y + z &= 9 \\ x + 2y - 3z &= -5 \\ 2x - 3y + 4z &= 11 \end{aligned}$$

Solution:

Starting with the Augmented Matrix

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 9 \\ 1 & 2 & -3 & -5 \\ 2 & -3 & 4 & 11 \end{array}\right]$$

we obtain

$$R_2 \leftrightarrow R_1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & -5 \\ 3 & -1 & 1 & 9 \\ 2 & -3 & 4 & 11 \end{array}\right]$$

$$\left.\begin{array}{l} -3R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array}\right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & -5 \\ 0 & -7 & 10 & 24 \\ 0 & -7 & 10 & 21 \end{array}\right]$$

Since $-7y + 10z = 24$ and $-7y + 10z = 21$ can NOT both be true, this system has NO solutions.

