Matrices
Systems of Linear Equations
(n by m: “n” equations with “m” unknowns)

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We will use the Reduced Row-Echelon form of the Augmented Matrix to solve a couple of n by m $(n \neq m)$ Linear System problems:

Example 01: Solve $\begin{matrix} 2x+3y-6z=10 \\ x+\ \ y-\ \ z=\ \ 2 \end{matrix}$ **;** 2 by 3 system

Solution:
We have

$$\left[\begin{array}{ccc|c} 2 & 3 & -6 & 10 \\ 1 & 1 & -1 & 2 \end{array}\right]$$

$$R_2 \leftrightarrow R_1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & 3 & -6 & 10 \end{array}\right]$$

$$\left.-2R_1+R_2 \to R_2\right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -4 & 6 \end{array}\right]$$

Since we are not going to get a unique solution, we will use the Row-Echelon Form to rewrite the system’s xyz form:

$$\begin{aligned} x+y-\ \ z&=2 \\ y-4z&=6 \Rightarrow y=4z+6 \end{aligned}$$

Thus

$$y=4z+6$$
$$\&$$
$$x+y-z=2 \Rightarrow x+(4z+6)-z=2$$
$$\Rightarrow x=2+z-4z-6=-3z-4$$

Setting $t=z$, ("t" is called the parameter), we have

$x = -3t - 4$
$y = 4t + 6$
$z = t$
OR
$(-3t - 4, 4t + 6, t)$

and we obtain an infinite number of solutions, on for each $t \in \mathbb{R}$.

Example 02: Solve $\begin{aligned} -x + 2y + 2z + 3w &= 4 \\ 3x + \ \ y + 4z + 2w &= 2 \\ x + \ \ y - \ \ z + 2w &= 3 \end{aligned}$; 3 by 4 system

Solution: Note: Calculations are tedious!
Starting with the Augmented Matrix

$$\left[\begin{array}{cccc|c} -1 & 2 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 & 2 \\ 1 & 1 & -1 & 2 & 3 \end{array}\right]$$

we obtain

$$-R_1 \to R_1 \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -2 & -3 & -4 \\ 3 & 1 & 4 & 2 & 2 \\ 1 & 1 & -1 & 2 & 3 \end{array}\right]$$

$$\left.\begin{array}{l} -3R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \end{array}\right\} \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -2 & -3 & -4 \\ 0 & 7 & 10 & 11 & 14 \\ 0 & 3 & 1 & 5 & 7 \end{array}\right]$$

$$\left.\frac{1}{7}R_2 \to R_2\right\} \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -2 & -3 & \frac{-4}{1} \\ 0 & 1 & \frac{10}{7} & \frac{11}{7} & \frac{2}{1} \\ 0 & 3 & 1 & 5 & \frac{7}{1} \end{array}\right]$$

$$\left.\begin{array}{l} 2R_2 + R_1 \to R_1 \\ -3R_2 + R_3 \to R_3 \end{array}\right\} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & \frac{6}{7} & \frac{1}{7} & \frac{0}{1} \\ 0 & 1 & \frac{10}{7} & \frac{11}{7} & \frac{2}{1} \\ 0 & 0 & -\frac{23}{7} & \frac{2}{7} & \frac{1}{1} \end{array}\right]$$

$$-\frac{7}{23}R_3 \to R_3 \Bigg\} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & \frac{6}{7} & \frac{1}{7} & \frac{0}{1} \\ 0 & 1 & \frac{10}{7} & \frac{11}{7} & \frac{2}{1} \\ 0 & 0 & 1 & -\frac{2}{23} & -\frac{7}{23} \end{array}\right]$$

$$\left.\begin{array}{l} -\frac{6}{7}R_3 + R_1 \to R_1 \\ -\frac{10}{7}R_3 + R_2 \to R_2 \end{array}\right\} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{5}{23} & \frac{6}{23} \\ 0 & 1 & 0 & \frac{39}{23} & \frac{56}{23} \\ 0 & 0 & 1 & -\frac{2}{23} & -\frac{7}{23} \end{array}\right]$$

Since we are not going to get a unique solution, we will use the Reduced Row-Echelon Form to rewrite the system's xyzw form:

$$\begin{array}{l} x + \frac{5w}{23} = \frac{6}{23} \Rightarrow x = -\frac{5w}{23} + \frac{6}{23} \Rightarrow x = -\frac{5t}{23} + \frac{6}{23} \\ y + \frac{39w}{23} = \frac{56}{23} \Rightarrow y = -\frac{39w}{23} + \frac{56}{23} \Rightarrow y = -\frac{39t}{23} + \frac{56}{23} \\ z - \frac{2w}{23} = -\frac{7}{23} \Rightarrow z = \frac{2w}{23} - \frac{7}{23} \Rightarrow z = \frac{2t}{23} - \frac{7}{23} \\ w = t \end{array}$$

where $t = w$, ("t" is called the parameter). Thus, there are an infinite number of solutions of the form $\left(-\frac{5t}{23} + \frac{6}{23}, -\frac{39t}{23} + \frac{56}{23}, \frac{2t}{23} - \frac{7}{23}, t\right)$, one for each $t \in \mathbb{R}$.

