Polynomial FUNctions
Division Algorithm

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Division Algorithm:

Given positive integers 127 and 4, we learned in arithmetic how to change the form of $\frac{127}{4}$ using long division:

$$\begin{array}{r} 31 \\ 4\overline{)127} \\ \underline{12} \\ 7 \\ \underline{4} \\ 3 \end{array} \qquad \begin{array}{r} \text{Quotient} \\ \text{Divisor}\overline{)\text{Dividend}} \\ \ldots \\ \underline{\hphantom{\text{Remainder}}} \\ \text{Remainder} \end{array}$$

must be < 4

so that

$$\frac{127}{4} = 31 + \frac{3}{4}\left(=31\frac{3}{4}\right) \quad \left\{\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}\right\}$$

$$\Rightarrow 127 = 31*4+3 \quad \{\text{Dividend} = \text{Quotient} * \text{Divisor} + \text{Remainder}\}$$

Note: $\text{Remainder} < \text{Divisor}$. This process is called the **Division Algorithm** for positive integers.

For polynomial functions, we have a corresponding algorithm:

Division Algorithm: Let $\mathbf{p}(\mathbf{x})$ & $\mathbf{d}(\mathbf{x})$ be polynomial functions with $\deg \mathbf{p}(\mathbf{x}) \geq \deg \mathbf{d}(\mathbf{x}) \geq 1$. Then there exist unique polynomials $\mathbf{q}(\mathbf{x})$ & $\mathbf{r}(\mathbf{x})$ such that

$$\mathbf{p}(\mathbf{x}) = \mathbf{q}(\mathbf{x}) * \mathbf{d}(\mathbf{x}) + \mathbf{r}(\mathbf{x})$$

$$\{\text{Dividend} = \text{Quotient} * \text{Divisor} + \text{Remainder}\}$$

where $\deg \mathbf{d}(\mathbf{x}) > \deg \mathbf{r}(\mathbf{x})$

Long division of polynomials is a key technique in applying the Division Algorithm:

In the following examples, given $\mathbf{p}(\mathbf{x})$ & $\mathbf{d}(\mathbf{x})$ find $\mathbf{q}(\mathbf{x})$ & $\mathbf{r}(\mathbf{x})$:

Example 01: $\mathbf{p}(\mathbf{x}) = \mathbf{x}^5 - 5\mathbf{x}^4 - \mathbf{x}^3 + 29\mathbf{x}^2 - 9\mathbf{x} - 34$ & $\mathbf{d}(\mathbf{x}) = \mathbf{x}^2 - 4$

Solution:

Using long division

$$\begin{array}{r|l}
 & \mathbf{x}^3 - 5\mathbf{x}^2 + 3\mathbf{x} + 9 \\
\hline
\mathbf{x}^2 + 0\mathbf{x} - 4 & \mathbf{x}^5 - 5\mathbf{x}^4 - \mathbf{x}^3 + 29\mathbf{x}^2 - 9\mathbf{x} - 34 \\
 & \underline{\mathbf{x}^5 \qquad - 4\mathbf{x}^3} \\
 & \quad - 5\mathbf{x}^4 + 3\mathbf{x}^3 + 29\mathbf{x}^2 - 9\mathbf{x} - 34 \\
 & \quad \underline{- 5\mathbf{x}^4 \qquad + 20\mathbf{x}^2} \\
 & \qquad 3\mathbf{x}^3 + 9\mathbf{x}^2 - 9\mathbf{x} - 34 \\
 & \qquad \underline{3\mathbf{x}^3 \qquad - 12\mathbf{x}} \\
 & \qquad\quad 9\mathbf{x}^2 + 3\mathbf{x} - 34 \\
 & \qquad\quad \underline{9\mathbf{x}^2 \qquad - 36} \\
 & \qquad\qquad 3\mathbf{x} \quad + 2
\end{array}$$

so that $\mathbf{q}(\mathbf{x}) = \mathbf{x}^3 - 5\mathbf{x}^2 + 3\mathbf{x} + 9$ & $\mathbf{r}(\mathbf{x}) = 3\mathbf{x} + 2$:

$$\frac{\mathbf{x}^5 - 5\mathbf{x}^4 - \mathbf{x}^3 + 29\mathbf{x}^2 - 9\mathbf{x} - 34}{\mathbf{x}^2 - 4} = \mathbf{x}^3 - 5\mathbf{x}^2 + 3\mathbf{x} + 9 + \frac{3\mathbf{x} + 2}{\mathbf{x}^2 - 4}$$

Example 02: $\mathbf{p}(\mathbf{x}) = \mathbf{x}^4 - 20\mathbf{x}^2 + \mathbf{x} + 67$ & $\mathbf{d}(\mathbf{x}) = \mathbf{x}^2 - 2\mathbf{x} - 8$

Solution:

Using long division

$$\begin{array}{r|l}
 & \mathbf{x}^2 + 2\mathbf{x} - 8 \\ \hline
\mathbf{x}^2 - 2\mathbf{x} - 8 & \mathbf{x}^4 + 0\mathbf{x}^3 - 20\mathbf{x}^2 + \mathbf{x} + 67 \\
 & \underline{\mathbf{x}^4 - 2\mathbf{x}^3 - 8\mathbf{x}^2} \\
 & 2\mathbf{x}^3 - 12\mathbf{x}^2 + \mathbf{x} + 67 \\
 & \underline{2\mathbf{x}^3 - 4\mathbf{x}^2 - 16\mathbf{x}} \\
 & -8\mathbf{x}^2 + 17\mathbf{x} + 67 \\
 & \underline{-8\mathbf{x}^2 + 16\mathbf{x} + 64} \\
 & \mathbf{x} + 3
\end{array}$$

so that $\mathbf{q}(\mathbf{x}) = \mathbf{x}^2 + 2\mathbf{x} - 8$ & $\mathbf{r}(\mathbf{x}) = \mathbf{x} + 3$

Example 03: $\mathbf{p(x)} = 2\mathbf{x}^3 + 11\mathbf{x}^2 + 3\mathbf{x} - 31$ & $\mathbf{d(x)} = \mathbf{x} + 4$

Solution:

Using long division

$$\begin{array}{r|l} & 2\mathbf{x}^2 + 3\mathbf{x} - 9 \\ \hline \mathbf{x}+4 & 2\mathbf{x}^3 + 11\mathbf{x}^2 + 3\mathbf{x} - 31 \\ & \underline{2\mathbf{x}^3 + 8\mathbf{x}^2} \\ & \qquad 3\mathbf{x}^2 + 3\mathbf{x} - 31 \\ & \qquad \underline{3\mathbf{x}^2 + 12\mathbf{x}} \\ & \qquad\qquad -9\mathbf{x} - 31 \\ & \qquad\qquad \underline{-9\mathbf{x} - 36} \\ & \qquad\qquad\qquad 5 \end{array}$$

so that $\mathbf{q(x)} = 2\mathbf{x}^2 + 3\mathbf{x} - 9$ & $\mathbf{r(x)} = 5$

It is bothersome to have to write and rewrite all the powers of x involved in the long division procedure (much less type them as I did)! The good news is that when deg d(x) = 1 (Linear), a procedure called **Synthetic Division** has been developed by analyzing the long division procedure to determine q(x) & r(x) more quickly.

Note: deg q(x) = deg p(x) – 1 ; r(x) = Constant

We will illustrate **Synthetic Division** by reworking the previous example. Note the color coding (you may not see the colors) of the step and **LD** stands for **Long Division**:

Previous Example: $\mathbf{p(x)} = 2\mathbf{x}^3 + 11\mathbf{x}^2 + 3\mathbf{x} - 31$ & $\mathbf{d(x)} = \mathbf{x} + 4$

$$\begin{array}{r|l} \overbrace{-4}^{\text{Opposite sign of "+4" in LD}} & 2 \quad 11 \quad 3 \quad -31 \quad \text{Coefficients of p(x) in decending powers} \\ & \qquad\qquad\qquad\quad \text{\& "0" for missing terms} \\ \hline & 2 \text{ Bring Down} \end{array}$$

$$\begin{array}{r|rrrr} -4 & 2 & 11 & 3 & -31 \\ & & -8 & & \\ \hline & 2 & & & \end{array}$$

Multiply (-4*2)

$$\begin{array}{r|rrrr} -4 & 2 & 11 & 3 & -31 \\ & & -8 & & \\ \hline & 2 & 3 & & \end{array}$$

Add

$$\begin{array}{r|rrrr} -4 & 2 & 11 & 3 & -31 \\ & & -8 & -12 & \\ \hline & 2 & 3 & & \end{array}$$

Multiply (-4*3)

$$\begin{array}{r|rrrr} -4 & 2 & 11 & 3 & -31 \\ & & -8 & -12 & \\ \hline & 2 & 3 & -9 & \end{array}$$

Add

$$\begin{array}{r|rrrr} -4 & 2 & 11 & 3 & -31 \\ & & -8 & -12 & 36 \\ \hline & 2 & 3 & -9 & \end{array}$$

Multiply (-4*(-9))

$$\begin{array}{r|rrrr} -4 & 2 & 11 & 3 & -31 \\ & & -8 & -12 & 36 \\ \hline & \underbrace{2 \quad 3 \quad -9}_{q(x)} & & & \underbrace{5}_{r(x)} \end{array}$$

Add

Thus $\mathbf{q}(\mathbf{x}) = 2\mathbf{x}^2 + 3\mathbf{x} - 9$ & $\mathbf{r}(\mathbf{x}) = 5$

Example 4: Given $\mathbf{p}(\mathbf{x}) = 3\mathbf{x}^4 + \mathbf{x}^3 - 13\mathbf{x}^2 - \mathbf{x} + 10$ & $\mathbf{d}(\mathbf{x}) = \mathbf{x} + 1$, use Synthetic Division to find $\mathbf{q}(\mathbf{x})$ & $\mathbf{r}(\mathbf{x})$.

Solution:

We have

$$\underbrace{-1}_{\text{Opposite sign of "+1"}} \quad \big) \; 3 \quad 1 \quad -13 \quad -1 \quad 10$$

$$\overline{3 \text{ Bring Down}}$$

$$
\begin{array}{r|rrrrr}
-1 & 3 & 1 & -13 & -1 & 10 \\
 & & -3 & & & \\
\hline
 & 3 & & & &
\end{array}
\quad \text{Multiply (-1*3)}
$$

$$
\begin{array}{r|rrrrr}
-1 & 3 & 1 & -13 & -1 & 10 \\
 & & -3 & & & \\
\hline
 & 3 & -2 & & &
\end{array}
\quad \text{Add}
$$

$$
\begin{array}{r|rrrrr}
-1 & 3 & 1 & -13 & -1 & 10 \\
 & & -3 & 2 & & \\
\hline
 & 3 & -2 & & &
\end{array}
\quad \text{Multiply (-1*(-2))}
$$

Rewritten:

$$\begin{array}{r|rrrrr} -1 & 3 & 1 & -13 & -1 & 10 \\ & & -3 & 2 & & \\ \hline & 3 & -2 & & & \end{array} \quad \text{Multiply (-1*(-2))}$$

$$\begin{array}{r|rrrrr} -1 & 3 & 1 & -13 & -1 & 10 \\ & & -3 & 2 & & \\ \hline & 3 & -2 & -11 & & \end{array} \quad \text{Add}$$

$$\begin{array}{r|rrrrr} -1 & 3 & 1 & -13 & -1 & 10 \\ & & -3 & 2 & 11 & \\ \hline & 3 & -2 & -11 & & \end{array} \quad \text{Multiply (-1*(-11))}$$

$$\begin{array}{r|rrrrr} -1 & 3 & 1 & -13 & -1 & 10 \\ & & -3 & 2 & 11 & \\ \hline & 3 & -2 & -11 & 10 & \end{array} \quad \text{Add}$$

$$\begin{array}{r|rrrrr} -1 & 3 & 1 & -13 & -1 & 10 \\ & & -3 & 2 & 11 & -10 \\ \hline & 3 & -2 & -11 & 10 & \end{array} \quad \text{Multiply (-1*10)}$$

$$\begin{array}{r|rrrrr} -1 & 3 & 1 & -13 & -1 & 10 \\ & & -3 & 2 & 11 & -10 \\ \hline & \underbrace{3 \quad -2 \quad -11 \quad 10}_{q(x)} & & & & \underbrace{0}_{r(x)} \end{array} \quad \text{Add}$$

Thus $\mathbf{q}(\mathbf{x}) = 3\mathbf{x}^3 - 2\mathbf{x}^2 - 11\mathbf{x} + 10$ & $\mathbf{r}(\mathbf{x}) = 0$

