Functions
Rational

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A **rational function** has the form

$$\begin{aligned} \mathbf{f}(\mathbf{x}) &= \frac{\text{Polynomial \#1}}{\text{Polyomial \#2}} \\ &= \frac{\mathbf{p}(\mathbf{x})}{\mathbf{q}(\mathbf{x})} \\ &= \frac{\mathbf{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_3 x^3 + a_2 x^2 + a_1 x + a_0}}{\mathbf{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_3 x^3 + b_2 x^2 + b_1 x + b_0}} \end{aligned}$$

Note: $\deg \mathbf{p} = \mathbf{n}$; $\deg \mathbf{q} = \mathbf{m}$ (Options: $\mathbf{n} > \mathbf{m}$; $\mathbf{n} = \mathbf{m}$; $\mathbf{n} < \mathbf{m}$)

We consider the first five (5) FUNction Summary Properties:

1. **Domain:**

   $\mathbf{Dom\ f} = \left\{ \mathbf{x} \in \mathbb{R} \mid \mathbf{q}(\mathbf{x}) \neq 0 \right\}_{\mathbf{x}}$

   Note: Maximum # of real numbers NOT in domain equals $\deg q = m$

2. **Intercepts:**

   y: Set $\mathbf{x} = 0 \Rightarrow (0, \mathbf{f}(0))$ if $0 \in \mathbf{Dom\ f}$

   x: Set $\mathbf{y} = \mathbf{f}(\mathbf{x}) = 0 \Rightarrow \left\{ (\mathbf{x}_0, 0) \mid \mathbf{p}(\mathbf{x}_0) = 0 \ \&\ \mathbf{q}(\mathbf{x}_0) \neq 0 \right\}$

3. **Continuity:**

   Note: $\mathbf{Cont\ f} = \mathbf{Dom\ f}$

   Note: If $\mathbf{x}_0 \in \mathbf{Discont\ f}$, then either

   a. $\mathbf{x} = \mathbf{x}_0$ is a vertical asymptote of $\mathbf{f}$
   b. there is a hole in the graph at $\mathbf{x}_0$

   Note: Rational functions do NOT have finite jumps!

   Note: Maximum # of holes and vertical asymptotes equals $\deg \mathbf{q} = \mathbf{m}$

$\mathbf{Pos\ f} = \left\{\mathbf{x} \in \mathbf{Dom\ f} \mid \mathbf{f}(\mathbf{x}) > 0\right\}_{\mathbf{x}}$: Graph of $\mathbf{f}$ is above the x-axis

$\mathbf{Neg\ f} = \left\{\mathbf{x} \in \mathbf{Dom\ f} \mid \mathbf{f}(\mathbf{x}) < 0\right\}_{\mathbf{x}}$: Graph of $\mathbf{f}$ is below the x-axis

Note: $\dfrac{\#}{\textbf{"Small"}} \Rightarrow \textbf{"Big"}\ ;\ \dfrac{\#}{\textbf{"Big"}} \Rightarrow \textbf{"Small"}$

4. **Behavior at/toward infinity:** $x \to \pm\infty$

$$\underset{\mathbf{x}\to -\infty}{\mathbf{Lim}}\ \mathbf{f}(\mathbf{x}) = \begin{cases} \mathbf{y}_0 \in \mathbb{R} & \left(\mathbf{y} = \mathbf{y}_0 \text{ is a horizontal asymptote})\right)\ ;\ (\mathbf{n} \le \mathbf{m}) \\ \pm\infty & (\mathbf{n} > \mathbf{m}) \end{cases}$$

$$\underset{\mathbf{x}\to -+\infty}{\mathbf{Lim}}\ \mathbf{f}(\mathbf{x}) = \begin{cases} \mathbf{y}_0 \in \mathbb{R} & \left(\mathbf{y} = \mathbf{y}_0 \text{ is a horizontal asymptote})\right)\ ;\ (\mathbf{n} \le \mathbf{m}) \\ \pm\infty & (\mathbf{n} > \mathbf{m}) \end{cases}$$

5. **Symmetry:**

Even: $\mathbf{f}(-\mathbf{x}) = \ldots = \mathbf{f}(\mathbf{x})$ (Graph symmetric wrt y-axis: $\mathbf{x} = 0$)

Odd: $\mathbf{f}(-\mathbf{x}) = \ldots = -\mathbf{f}(\mathbf{x})$ (Graph symmetric wrt origin: $(0,0)$)

Important Notes:

1. $\dfrac{\#}{\text{"BIG"}} = \text{"SMALL"}$
2. $\dfrac{\#}{\text{"SMALL"}} = \text{"BIG"}$
3. $\dfrac{0}{0}$ = Indeterminate Form (Haven't done the right thing YET!)
4. $\dfrac{\pm\infty}{\pm\infty}$ = Indeterminate Form (Haven't done the right thing YET!)

5. Important Table

x values	x pattern	f(x) pattern
$x \to x_0^-$ (Left)	x →→→→ //////////x_0^- __________	$f(x) \to ?$
$x \to x_0^+$ (Right)	←←←← x __________ x_0^+\\\\\\\\\\	$f(x) \to ?$
$x \to -\infty$ $(-\infty \leftarrow x)$	←←←← x __0__________	$f(x) \to ?$
$x \to +\infty$	x →→→→ _____0__//////////	$f(x) \to ?$

Example 01:

Find the first five (5) FUNction Summary Properties of $\mathbf{f(x)} = \frac{2\mathbf{x}^2}{\mathbf{x}^2+4}$ and sketch its graph (best we can without the calculus)

Solution:

1. **Domain:** $\mathbf{Dom\ f} = \mathbb{R}_\mathbf{x}$
2. **Intercepts:** $(0,0)$ is an x and y intercept POINT
3. **Continuity:**
   $\mathbf{Cont\ f} = \mathbf{Dom\ f} = \mathbb{R}_\mathbf{x}$

   $\overbrace{\qquad\qquad\qquad}^{+}\ 0\ \overbrace{\qquad\qquad\qquad}^{+}$ Sign of f

   $\mathbf{Pos\ f} = (-\infty,0)_\mathbf{x} \cup (0,+\infty)_\mathbf{x}$

   $\mathbf{Neg\ f} = \phi$; Empty set
4. **Behavior at/toward Infinity:**

   $$\lim_{\mathbf{x}\to-\infty} \mathbf{f(x)} = \lim_{\mathbf{x}\to-\infty} \frac{2\mathbf{x}^2}{\mathbf{x}^2+4} = \lim_{\mathbf{x}\to-\infty} \frac{2}{1+\frac{4}{\mathbf{x}^2}} = 2$$

   (Divide numerator & denominator by highest power: $\mathbf{x}^2$)

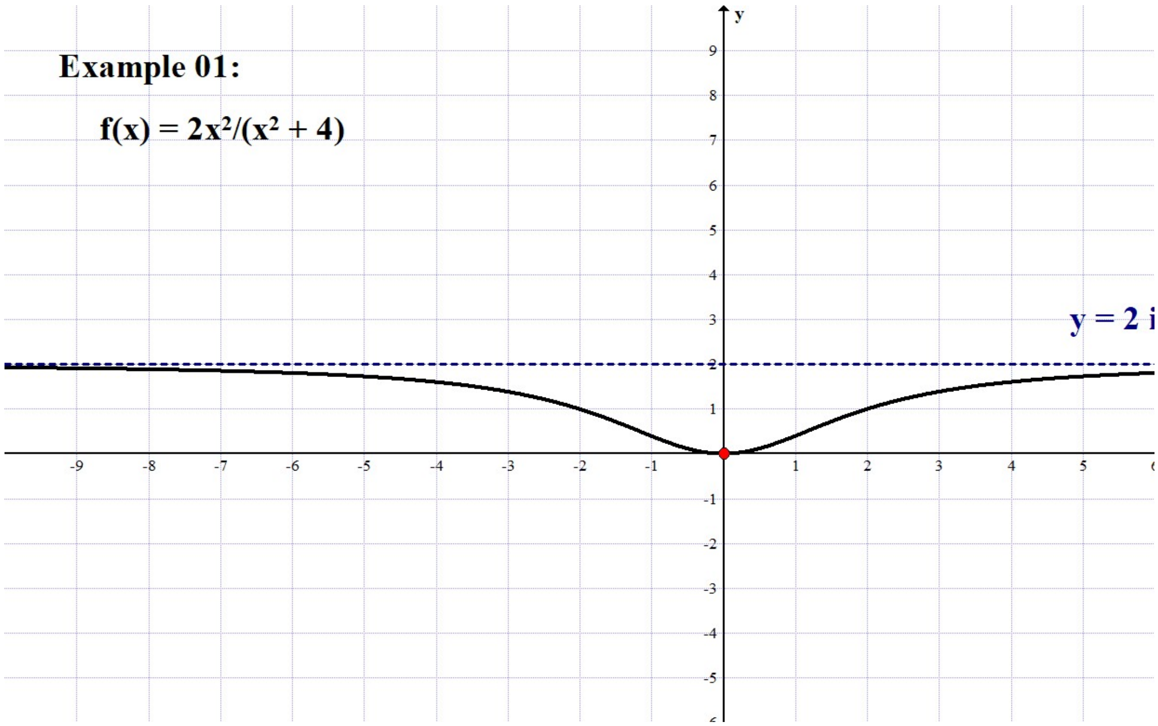
   Note: $\left(\frac{4}{\mathbf{Big}} \to 0\right)$; $\left(\frac{\mathbf{Num}=2}{\mathbf{Den}\approx 1} \to 2\right)$

   $\therefore \mathbf{y} = 2$ is a horizontal asymptote

   $\lim_{\mathbf{x}\to+\infty} \mathbf{f(x)} = 2$
5. **Symmetry:**

   $$\mathbf{f}(-\mathbf{x}) = \frac{2(-\mathbf{x})^2}{(-\mathbf{x})^2+4} = \frac{2\mathbf{x}^2}{\mathbf{x}^2+4} = \mathbf{f}(\mathbf{x}) \Rightarrow \text{Even (Graph symmetric wrt } \mathbf{x}=0)$$

6. Graph what the information above yields and then look at the actual graph on the next page.


Example 01:
f(x) = 2x²/(x² + 4)
y = 2 i
y
9
8
7
6
5
4
3
2
1
-1
-2
-3
-4
-5
-9
-8
-7
-6
-5
-4
-3
-2
-1
1
2
3
4
5

Example 2:

Find the first five (5) FUNction Summary Properties of $\mathbf{f}(\mathbf{x})=\dfrac{\mathbf{x}+2}{\mathbf{x}^3-4\mathbf{x}}$ and sketch its graph (best we can without the calculus)

Solution:

1. **Domain:**

   $\mathbf{x}^3-4\mathbf{x}\overset{\text{SET}}{=}0 \Rightarrow \mathbf{x}=-2,0,2$

   $\textbf{Dom f}=\mathbb{R}_{\mathbf{x}}\setminus\{-2,0,2\}_{\mathbf{x}}$

   Note: $\mathbf{f}(\mathbf{x})=\dfrac{\mathbf{x}+2}{\mathbf{x}^3-4\mathbf{x}}=\dfrac{\mathbf{x}+2}{\mathbf{x}(\mathbf{x}-2)(\mathbf{x}+2)}\underset{\mathbf{x}\neq-2}{=}\dfrac{1}{\mathbf{x}(\mathbf{x}-2)}$

2. **Intercepts:** There are NO y or x intercept POINTS since $-2\notin\textbf{Dom f}$

3. **Continuity:**

   $\textbf{Cont f}=\textbf{Dom f}=\mathbb{R}_{\mathbf{x}}\setminus\{-2,0,2\}_{\mathbf{x}}$

   $\overbrace{\rule{2cm}{0.4pt}}^{+}\,-2\,\overbrace{\rule{2cm}{0.4pt}}^{+}\,0\,\overbrace{\rule{2cm}{0.4pt}}^{-}\,2\,\overbrace{\rule{2cm}{0.4pt}}^{+}$ Sign of f

   $\textbf{Pos f}=(-\infty,-2)_{\mathbf{x}}\cup(-2,0)_{\mathbf{x}}\cup(2,+\infty)_{\mathbf{x}}$

   $\textbf{Neg f}=(0,2)_{\mathbf{x}}$

   Discontinuity : $\mathbf{x}=-2$:

   Since $-2\notin\textbf{Dom f}$, we try to find out what is happening to the $\mathbf{f}(\mathbf{x})$ values for $\mathbf{x}$ values "close to" but unequal to -2.

   From the left: $\mathbf{x}\to-2^-\Rightarrow\mathbf{f}(\mathbf{x})\to\dfrac{1}{8}\left(\mathbf{f}(\mathbf{x})=\dfrac{1}{\mathbf{x}(\mathbf{x}-2)}\underset{\mathbf{x}\to-2^-}{\Rightarrow}\dfrac{=1}{\approx 8}\right)$

   Also written $\displaystyle\lim_{\mathbf{x}\to-2^-}\mathbf{f}(\mathbf{x})=\frac{1}{8}$

   From the right: $\mathbf{x}\to-2^+\Rightarrow\mathbf{f}(\mathbf{x})\to\dfrac{1}{8}\left(\mathbf{f}(\mathbf{x})=\dfrac{1}{\mathbf{x}(\mathbf{x}-2)}\underset{\mathbf{x}\to-2^+}{\Rightarrow}\dfrac{=1}{\approx 8}\right)$

   Also written $\displaystyle\lim_{\mathbf{x}\to-2^+}\mathbf{f}(\mathbf{x})=\frac{1}{8}$

   $\therefore$ Hole: $\left(-2,\dfrac{1}{8}\right)$

Discontinuity : $\mathbf{x} = 0$:

Since $0 \notin \mathbf{Dom\ f}$, we try to find out what is happening to the $\mathbf{f}(\mathbf{x})$ values for $\mathbf{x}$ values "close to" but unequal to 0.

From the left: $\mathbf{x} \to 0^- \Rightarrow \mathbf{f}(\mathbf{x}) \to +\infty \left(\mathbf{f}(\mathbf{x}) = \dfrac{1}{\mathbf{x}(\mathbf{x}-2)} \underset{\mathbf{x}\to 0^-}{\Rightarrow} \dfrac{=1}{\approx 0(+)} \right)$

Also wirtten $\lim\limits_{\mathbf{x}\to 0^-} \mathbf{f}(\mathbf{x}) = +\infty$

$\therefore$ Vertical Asymptote: $\mathbf{x} = 0$

From the right: $\mathbf{x} \to 0^+ \Rightarrow \mathbf{f}(\mathbf{x}) \to -\infty \left(\mathbf{f}(\mathbf{x}) = \dfrac{1}{\mathbf{x}(\mathbf{x}-2)} \underset{\mathbf{x}\to 0^+}{\Rightarrow} \dfrac{=1}{\approx 0(-)} \right)$

Also written $\lim\limits_{\mathbf{x}\to 0^+} \mathbf{f}(\mathbf{x}) = -\infty$

Discontinuity : $\mathbf{x} = 2$:

Since $2 \notin \mathbf{Dom\ f}$, we try to find out what is happening to the $\mathbf{f}(\mathbf{x})$ values for $\mathbf{x}$ values "close to" but unequal to 2.

From the left: $\mathbf{x} \to 2^- \Rightarrow \mathbf{f}(\mathbf{x}) \to -\infty \left(\mathbf{f}(\mathbf{x}) = \dfrac{1}{\mathbf{x}(\mathbf{x}-2)} \underset{\mathbf{x}\to 2^-}{\Rightarrow} \dfrac{=1}{\approx 0(-)} \right)$

Also written $\lim\limits_{\mathbf{x}\to 2^-} \mathbf{f}(\mathbf{x}) = -\infty$

$\therefore$ Vertical Asymptote: $\mathbf{x} = 2$

From the right: $\mathbf{x} \to 2^+ \Rightarrow \mathbf{f}(\mathbf{x}) \to +\infty \left(\mathbf{f}(\mathbf{x}) = \dfrac{1}{\mathbf{x}(\mathbf{x}-2)} \underset{\mathbf{x}\to 2^+}{\Rightarrow} \dfrac{=1}{\approx 0(+)} \right)$

Also written $\lim\limits_{\mathbf{x}\to 2^+} \mathbf{f}(\mathbf{x}) = +\infty$

4. **Behavior at/toward Infinity:**

$$\lim_{\mathbf{x}\to -\infty} \mathbf{f}(\mathbf{x}) = \lim_{\mathbf{x}\to -\infty} \frac{\mathbf{x}+2}{\mathbf{x}^3 - 4\mathbf{x}} = \lim_{\mathbf{x}\to -\infty} \frac{\frac{1}{\mathbf{x}^2} + \frac{2}{\mathbf{x}^3}}{1 - \frac{4}{\mathbf{x}^2}} = 0$$

(Divide numerator & denominator by highest power: $\mathbf{x}^3$)

Note: $\left(\dfrac{1}{\mathbf{x}^2}, \dfrac{2}{\mathbf{x}^3}, \dfrac{4}{\mathbf{x}^2} \Rightarrow \dfrac{=\#}{\mathbf{Big}} \to 0 \right); \left(\dfrac{\mathbf{Num} \approx 0}{\mathbf{Den} \approx 1} \to 0 \right)$

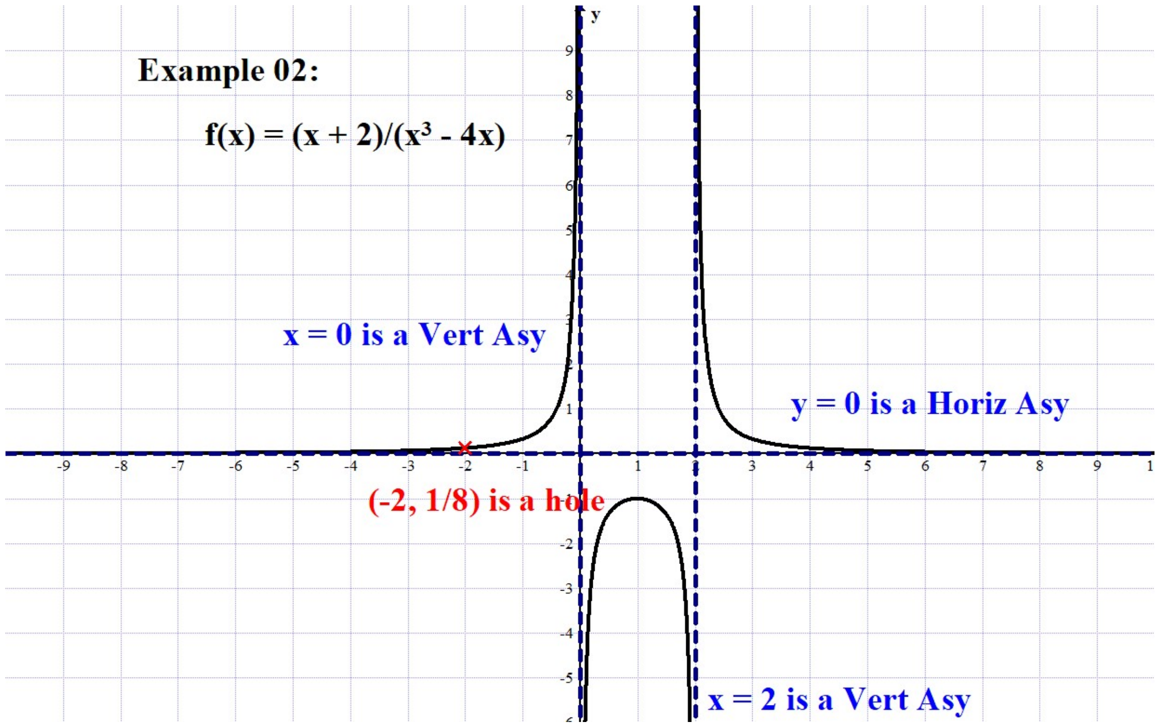
$\therefore \mathbf{y} = 0$ is a horizontal asymptote

$\lim\limits_{\mathbf{x}\to +\infty} \mathbf{f}(\mathbf{x}) = 0$

5. **Symmetry:**

$$\mathbf{f}(-\mathbf{x}) = \frac{(-\mathbf{x})+2}{(-\mathbf{x})^3 - 4(-\mathbf{x})} = \frac{\mathbf{x}-2}{\mathbf{x}^3 - 4\mathbf{x}} \neq \begin{cases} \mathbf{f}(\mathbf{x}) \\ -\mathbf{f}(\mathbf{x}) \end{cases} \Rightarrow \text{Neither even/odd}$$

6. Graph what the information above yields and then look at the actual graph on the next page.


Example 02:
f(x) = (x + 2)/(x³ - 4x)
x = 0 is a Vert Asy
y = 0 is a Horiz Asy
(-2, 1/8) is a hole
x = 2 is a Vert Asy

Example 3:

Find the first five (5) FUNction Summary Properties of $\mathbf{f}(\mathbf{x})=\dfrac{\mathbf{x}^2-4}{\mathbf{x}^2-6\mathbf{x}-27}$ and sketch its graph (best we can without the calculus)

Solution:

1. **Domain:**

   $\mathbf{x}^2-6\mathbf{x}-27\overset{\text{SET}}{=}0 \Rightarrow (\mathbf{x}+3)(\mathbf{x}-9)=0 \Rightarrow \mathbf{x}=-3,9$

   $\textbf{Dom f}=\mathbb{R}_{\mathbf{x}}\setminus\{-3,9\}_{\mathbf{x}}$

2. **Intercepts:**

   y-intercept POINT: $\left(0,\dfrac{4}{27}\right)$; Set $\mathbf{x}=0$; Evaluate

   x-intercept POINT:

   $(-2,0)$; $(+2,0)$; Set $\mathbf{f}(\mathbf{x})=\mathbf{x}^2-4=(\mathbf{x}+2)(\mathbf{x}-2)\overset{\text{SET}}{=}0$; Solve

3. **Continuity:**

   $\textbf{Cont f}=\textbf{Dom f}=\mathbb{R}_{\mathbf{x}}\setminus\{-3,9\}_{\mathbf{x}}$; same as domain for rational functions

   Note: $\mathbf{f}(\mathbf{x})=\dfrac{\mathbf{x}^2-4}{\mathbf{x}^2-6\mathbf{x}-27}=\dfrac{(\mathbf{x}-2)(\mathbf{x}+2)}{(\mathbf{x}-9)(\mathbf{x}+3)}$

   $\overbrace{\rule{3em}{0.4pt}}^{+}-3\overbrace{\rule{2em}{0.4pt}}^{-}-2\overbrace{\rule{3em}{0.4pt}}^{+}2\overbrace{\rule{3em}{0.4pt}}^{-}9\overbrace{\rule{3em}{0.4pt}}^{+}$ Sign of f

   $\textbf{Pos f}=(-\infty,-3)_{\mathbf{x}}\cup(-3,2)_{\mathbf{x}}\cup(9,+\infty)_{\mathbf{x}}$

   $\textbf{Neg f}=(-3,-2)_{\mathbf{x}}\cup(2,9)_{\mathbf{x}}$

   Discontinuity : $\mathbf{x}=-3$:

   Since $-3\notin$ **Dom f**, we try to find out what is happening to the $\mathbf{f}(\mathbf{x})$ values for $\mathbf{x}$ values "close to" but unequal to -3.

   From the left: $\mathbf{x}\to-3^- \Rightarrow \mathbf{f}(\mathbf{x})\to+\infty$ $\left(\mathbf{f}(\mathbf{x})=\dfrac{(\mathbf{x}-2)(\mathbf{x}+2)}{(\mathbf{x}+3)(\mathbf{x}-9)}\underset{\mathbf{x}\to-3^-}{\Rightarrow}\dfrac{\approx 5}{\approx 0(+)}\right)$

   Also written $\lim\limits_{\mathbf{x}\to-3^-}\mathbf{f}(\mathbf{x})=+\infty$

   $\therefore$ Vertical Asymptote: $\mathbf{x}=-3$

   From the right: $\mathbf{x}\to-3^+ \Rightarrow \mathbf{f}(\mathbf{x})\to-\infty$ $\left(\mathbf{f}(\mathbf{x})\underset{\mathbf{x}\to-3^+}{\Rightarrow}\dfrac{\approx 5}{\approx 0(-)}\right)$

   Also written $\lim\limits_{\mathbf{x}\to-3^+}\mathbf{f}(\mathbf{x})=-\infty$

Discontinuity : $\mathbf{x}=9$:

Since $9 \notin \mathbf{Dom\ f}$, we try to find out what is happening to the $\mathbf{f}(\mathbf{x})$ values for $\mathbf{x}$ values "close to" but unequal to 9.

From the left: $\mathbf{x} \to 9^- \Rightarrow \mathbf{f}(\mathbf{x}) \to -\infty \left(\mathbf{f}(\mathbf{x}) = \dfrac{(\mathbf{x}-2)(\mathbf{x}+2)}{(\mathbf{x}+3)(\mathbf{x}-9)} \underset{\mathbf{x}\to 9^-}{\Rightarrow} \dfrac{\approx 77}{\approx 0(-)} \right)$

Also written $\lim\limits_{\mathbf{x}\to 9^-} \mathbf{f}(\mathbf{x}) = -\infty$

Vertical Asymptote: $\mathbf{x}=9$

From the right: $\mathbf{x} \to 9^+ \Rightarrow \mathbf{f}(\mathbf{x}) \to +\infty \left(\mathbf{f}(\mathbf{x}) \underset{\mathbf{x}\to 9^+}{\Rightarrow} \dfrac{\approx 77}{\approx 0(+)} \right)$

Also written $\lim\limits_{\mathbf{x}\to 9^+} \mathbf{f}(\mathbf{x}) = +\infty$

4. **Behavior at/toward Infinity:**

$$\lim_{\mathbf{x}\to-\infty} \mathbf{f}(\mathbf{x}) = \lim_{\mathbf{x}\to-\infty} \frac{\mathbf{x}^2-4}{\mathbf{x}^2-6\mathbf{x}-27} = \lim_{\mathbf{x}\to-\infty} \frac{1-\frac{4}{\mathbf{x}^2}}{1-\frac{6}{\mathbf{x}}-\frac{27}{\mathbf{x}^2}} = 1$$

(Divide numerator & denominator by highest power: $\mathbf{x}^2$

Note: $\left(\dfrac{4}{\mathbf{x}^2}, \dfrac{6}{\mathbf{x}}, \dfrac{27}{\mathbf{x}^3} \Rightarrow \dfrac{\#}{\mathbf{Big}} \to 0 \right); \left(\dfrac{\mathbf{Num} \to 1}{\mathbf{Den} \to 1} \to 1 \right)$

$\therefore \mathbf{y}=1$ is a horizontal asymptote

$\lim\limits_{\mathbf{x}\to+\infty} \mathbf{f}(\mathbf{x}) = 1$

If there is an x-value such that $\mathbf{f}(\mathbf{x})=1$, then then graph and Horizontal Asymptote intersect:

$$\mathbf{f}(\mathbf{x}) \overset{\text{SET}}{=} 1 \Rightarrow \frac{\mathbf{x}^2-4}{\mathbf{x}^2-6\mathbf{x}-27} = 1$$

$$\Rightarrow \mathbf{x}^2-4 = \mathbf{x}^2-6\mathbf{x}-27$$

$$\Rightarrow -4 = -6\mathbf{x}-27$$

$$\Rightarrow 6\mathbf{x} = -27+4 = -23$$

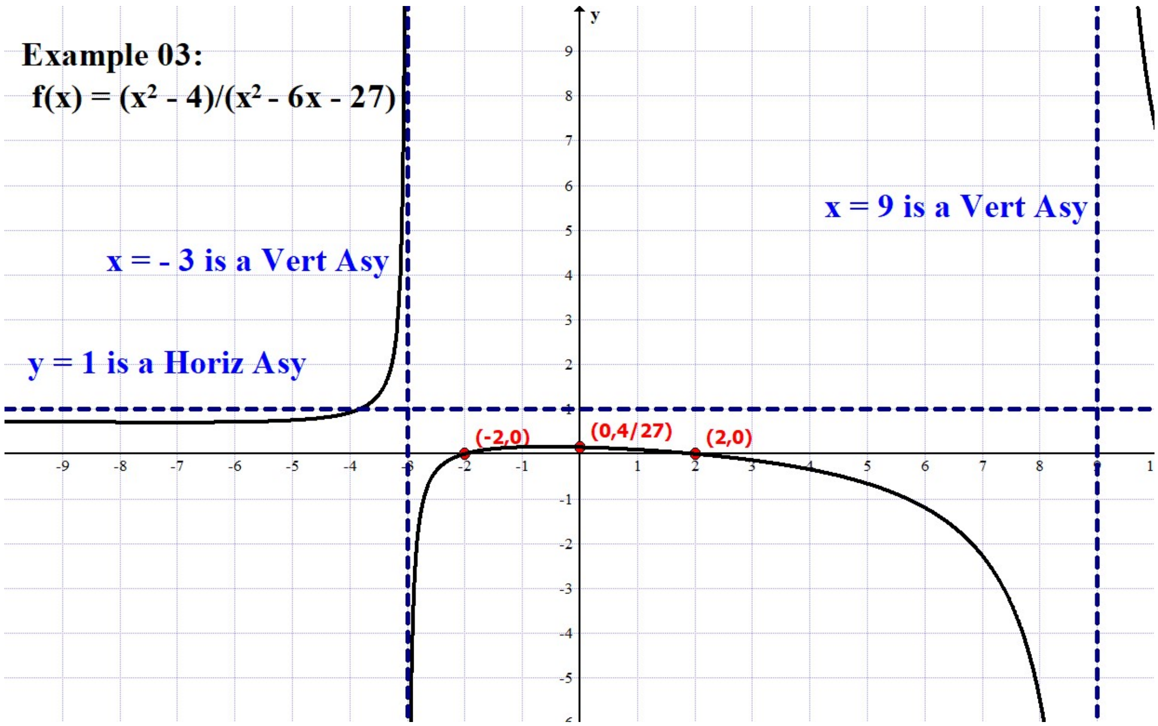
$$\Rightarrow \mathbf{x} = -\frac{23}{6} = -3\frac{5}{6}$$

$\Rightarrow$ Graph intersects Horiz Asy at $\left(-\dfrac{23}{6}, 1 \right)$

5. **Symmetry:**

$$\mathbf{f}(-\mathbf{x}) = \frac{(-\mathbf{x})^2-4}{(-\mathbf{x})^2-6(-\mathbf{x})-27} = \frac{\mathbf{x}^2-4}{\mathbf{x}^2+6\mathbf{x}-27} \neq \begin{cases} \mathbf{f}(\mathbf{x}) \\ -\mathbf{f}(\mathbf{x}) \end{cases} \Rightarrow \text{Neither even/odd}$$

6. Graph what the information above yields and then look at the actual graph on the next page.



Example 4:

Find the first five (5) FUNction Summary Properties of $\mathbf{f}(\mathbf{x})=\frac{\mathbf{x}^2-1}{\mathbf{x}-1}$ and sketch its graph (best we can without the calculus)

Solution:

1. **Domain:**

   $\mathbf{Dom\ f}=\mathbb{R}_{\mathbf{x}}\setminus\{1\}_{\mathbf{x}}$

   $\mathbf{f}(\mathbf{x})=\frac{\mathbf{x}^2-1}{\mathbf{x}-1}\underset{\mathbf{x}\neq 1}{=}\mathbf{x}+1$

2. **Intercepts:**

   y-intercept POINT: $(0,1)$

   x-intercept POINT: $(-1,0)$ since $1\notin$ **Dom f**

3. **Continuity:**

   $\mathbf{Cont\ f}=\mathbf{Dom\ f}=\mathbb{R}_{\mathbf{x}}\setminus\{1\}_{\mathbf{x}}$

   $$\overbrace{\_\_\_\_\_\_\_\_}^{-}\,-1\,\overbrace{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}^{+}\,1\,\overbrace{\_\_\_\_\_\_\_\_}^{+}\ \text{Sign of f}$$

   $\mathbf{Pos\ f}=(-1,1)_{\mathbf{x}}\cup(1,+\infty)_{\mathbf{x}}$

   $\mathbf{Neg\ f}=(-\infty,-1)_{\mathbf{x}}$

   Disconiutity $:\mathbf{x}=1$

   Since $1\notin$ **Dom f**, we try to find out what is happening to the $\mathbf{f}(\mathbf{x})$ values for $\mathbf{x}$ values "close to" but unequal to 1.

   From the left: $\mathbf{x}\to 1^- \Rightarrow \mathbf{f}(\mathbf{x})\to 2 \left(\mathbf{f}(\mathbf{x})=\mathbf{x}+1 \underset{\mathbf{x}\to 1^-}{\Rightarrow}\approx 2\right)$

   Also written $\underset{\mathbf{x}\to 1^-}{\mathbf{Lim}}\ \mathbf{f}(\mathbf{x})=2$

   From the right: $\mathbf{x}\to 1^+ \Rightarrow \mathbf{f}(\mathbf{x})\to 2 \left(\mathbf{f}(\mathbf{x}) \underset{\mathbf{x}\to 1^+}{\Rightarrow}\approx 2\right)$

   Also written $\underset{\mathbf{x}\to 1^+}{\mathbf{Lim}}\ \mathbf{f}(\mathbf{x})=2$

   $\therefore$ Hole: $(1,2)$

4. **Behavior at/toward Infinity:**

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2-1}{x-1} = \lim_{x\to-\infty} \frac{1-\frac{1}{x^2}}{\frac{1}{x}-\frac{1}{x^2}} = -\infty$$

(Divide numerator & denominator by highest power: x^2)

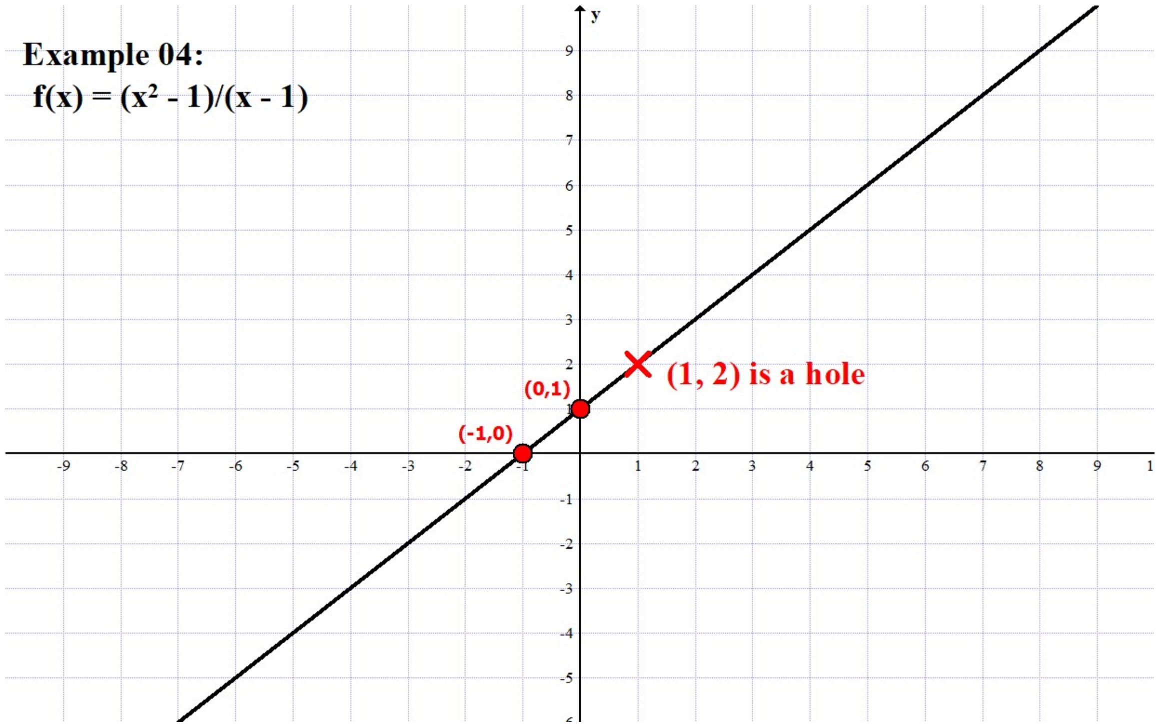
$$\text{Note:}\left(\frac{1}{x^2},\frac{1}{x} \Rightarrow \frac{=1}{\approx \textbf{Big}} \to 0\right);\left(\frac{\textbf{Numer} \to 1}{\textbf{Denom} \to 0(-)} \to -\infty\right)$$

$$\lim_{x\to+\infty} f(x) = +\infty$$

5. **Symmetry:**

$$f(-x) = \frac{(-x)^2-1}{(-x)-1} = -\frac{x^2-1}{x+1} \neq \begin{cases} f(x) \\ -f(x) \end{cases} \Rightarrow \text{Neither even/odd}$$

6. Graph what the information above yields and then look at the actual graph on the next page.


Example 04:
f(x) = (x² - 1)/(x - 1)
y
(1, 2) is a hole
(0,1)
(-1,0)

Example 5:

Find the first five (5) FUNction Summary Properties of $\mathbf{f}(\mathbf{x})=\dfrac{\mathbf{x}^2+4}{\mathbf{x}}$ and sketch its graph (best we can without the calculus)

Solution:

1. **Domain:**
   $\textbf{Dom f}=\mathbb{R}_\mathbf{x}\setminus\{0\}_\mathbf{x}$
2. **Intercepts:**
   There are NO y or x intercept POINTS
3. **Continuity:**
   $\textbf{Cont f}=\textbf{Dom f}=\mathbb{R}_\mathbf{x}\setminus\{0\}_\mathbf{x}$

   $$\overbrace{\underline{\qquad\qquad\qquad\qquad}}^{-}\,0\,\overbrace{\underline{\qquad\qquad\qquad\qquad}}^{+}\ \text{Sign of f}$$

   $\textbf{Pos f}=(0,+\infty)_\mathbf{x}$

   $\textbf{Neg f}=(-\infty,0)_\mathbf{x}$

   Discontinuity : $\mathbf{x}=0$

   Since $0\notin\textbf{Dom f}$, we try to find out what is happening to the $\mathbf{f}(\mathbf{x})$ values for $\mathbf{x}$ values "close to" but unequal to 0.

   From the left: $\mathbf{x}\to 0^-\Rightarrow \mathbf{f}(\mathbf{x})\to-\infty\left(\mathbf{f}(\mathbf{x})=\dfrac{\mathbf{x}^2+4}{\mathbf{x}}\underset{\mathbf{x}\to0^-}{\Rightarrow}\dfrac{\approx 4}{\approx 0(-)}\right)$

   Also written $\lim\limits_{\mathbf{x}\to 0^-}\mathbf{f}(\mathbf{x})=-\infty$

   $\therefore$ Vertical Asymptote: $\mathbf{x}=0$

   From the right: $\mathbf{x}\to 0^+\Rightarrow \mathbf{f}(\mathbf{x})\to+\infty\left(\mathbf{f}(\mathbf{x})\underset{\mathbf{x}\to0^+}{\Rightarrow}\dfrac{\approx 4}{\approx 0(+)}\right)$

   Also written: $\lim\limits_{\mathbf{x}\to 0^+}\mathbf{f}(\mathbf{x})=+\infty$

4. **Behavior at/toward Infinity:**

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2+4}{x} = \lim_{x\to-\infty} \frac{1+\frac{4}{x^2}}{\frac{1}{x}} = -\infty$$

(Divide numerator & denominator by highest power: x^2)

Note: $\left(\frac{4}{x^2}, \frac{1}{x} \Rightarrow \frac{\#}{\text{Big}} \to 0\right); \left(\frac{\text{Numer} \to 1}{\text{Denom} \to 0(-)} \to -\infty\right)$

$$\lim_{x\to+\infty} f(x) = +\infty$$

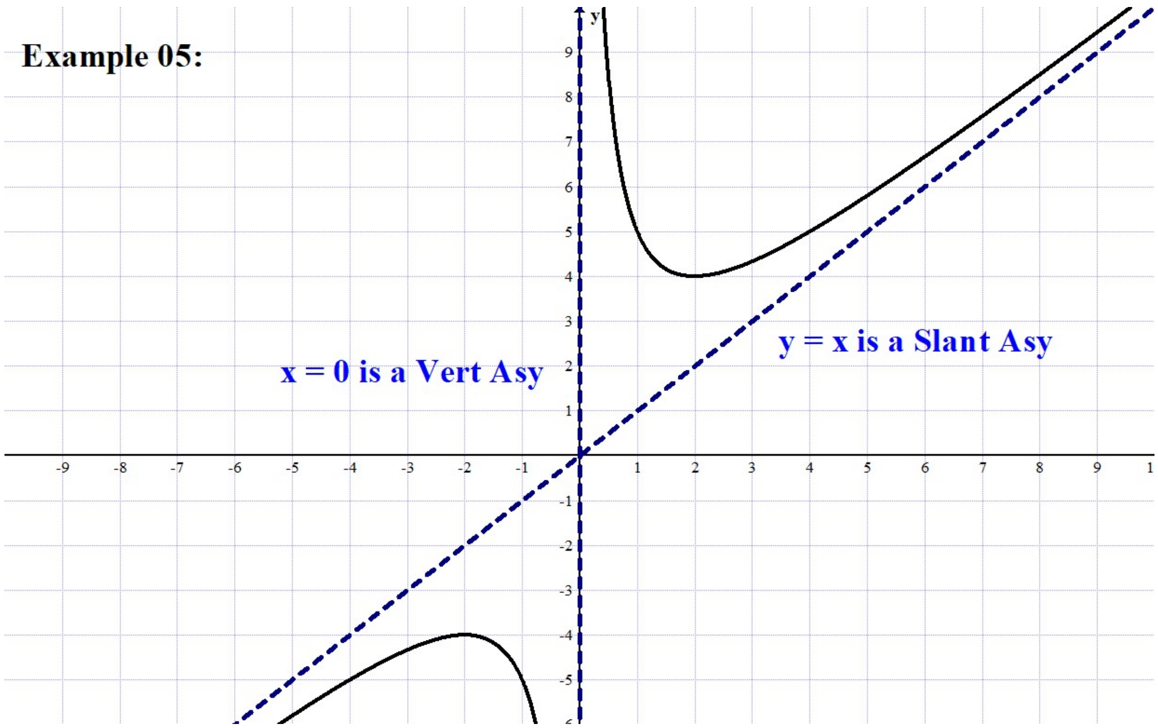
$f(x) = \frac{x^2+4}{x} = x + \frac{4}{x} \approx x$ when x is BIG!

$\Rightarrow y = x$ is a Slant Asymptote

5. **Symmetry:**

$f(-x) = \frac{(-x)^2+4}{(-x)} = -\frac{x^2+4}{x} = -f(x) \Rightarrow$ Odd (Graph symmetric wrt (0,0))

6. Graph what the information above yields and then look at the actual graph on the next page.


Example 05:
y
x = 0 is a Vert Asy
y = x is a Slant Asy