

Functions

Rational

MATH by Wilson
Your Personal Mathematics Trainer
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A **rational function** has the form

$$\begin{aligned}f(x) &= \frac{\text{Polynomial \#1}}{\text{Polynomial \#2}} \\ &= \frac{p(x)}{q(x)} \\ &= \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_3 x^3 + a_2 x^2 + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_3 x^3 + b_2 x^2 + b_1 x + b_0}\end{aligned}$$

Note: $\deg p = n$; $\deg q = m$ (Options: $n > m$; $n = m$; $n < m$)

We consider the first five (5) FUNction Summary Properties:

1. **Domain:**

$$\text{Dom } f = \{x \in \mathbb{R} \mid q(x) \neq 0\}_x$$

Note: Maximum # of real numbers NOT in domain equals $\deg q = m$

2. **Intercepts:**

y: Set $x = 0 \Rightarrow (0, f(0))$ if $0 \in \text{Dom } f$

x: Set $y = f(x) = 0 \Rightarrow \{(x_0, 0) \mid p(x_0) = 0 \ \& \ q(x_0) \neq 0\}$

3. **Continuity:**

Note: $\text{Cont } f = \text{Dom } f$

Note: If $x_0 \in \text{Discont } f$, then either

- $x = x_0$ is a vertical asymptote of **f** **IF** $(x - x_0)$ is a factor of $q(x)$ but NOT $p(x)$

- b. there is a hole in the graph at x_0 **IF** *each* $(x - x_0)$ is a factor of both $p(x)$ & $q(x)$

Note: Rational functions do NOT have finite jumps!

Note: Maximum # of holes and vertical asymptotes equals $\deg q = m$

Pos $f = \{x \in \text{Dom } f \mid f(x) > 0\}_x$: Graph of f is above the x-axis

Neg $f = \{x \in \text{Dom } f \mid f(x) < 0\}_x$: Graph of f is below the x-axis

Note: $\frac{\#}{\text{"Small"}} \Rightarrow \text{"Big"} ; \frac{\#}{\text{"Big"}} \Rightarrow \text{"Small"}$

4. **Behavior at/toward infinity:** $x \rightarrow \pm \infty$

- $n < m$: $x \rightarrow \pm \infty \Rightarrow f(x) \rightarrow 0$; $y = 0$ is a horizontal asymptote
- $n = m$: $x \rightarrow \pm \infty \Rightarrow f(x) \rightarrow \frac{a_n}{b_m}$; $y = \frac{a_n}{b_m}$ is a horizontal asymptote
- $n > m$: $x \rightarrow \pm \infty \Rightarrow$ either $f(x) \rightarrow -\infty$ OR $f(x) \rightarrow +\infty$; more analysis required

Calculus related: Omit for now

$$\lim_{x \rightarrow -\infty} f(x) = \begin{cases} y_0 \in \mathbb{R} & (y = y_0 \text{ is a horizontal asymptote}) ; (n \leq m) \\ \pm \infty & (n > m) \end{cases}$$

$$\lim_{x \rightarrow +\infty} f(x) = \begin{cases} y_0 \in \mathbb{R} & (y = y_0 \text{ is a horizontal asymptote}) ; (n \leq m) \\ \pm \infty & (n > m) \end{cases}$$

5. **Symmetry:**

Even: $f(-x) = \dots = f(x)$ (Graph symmetric wrt y-axis: $x = 0$)

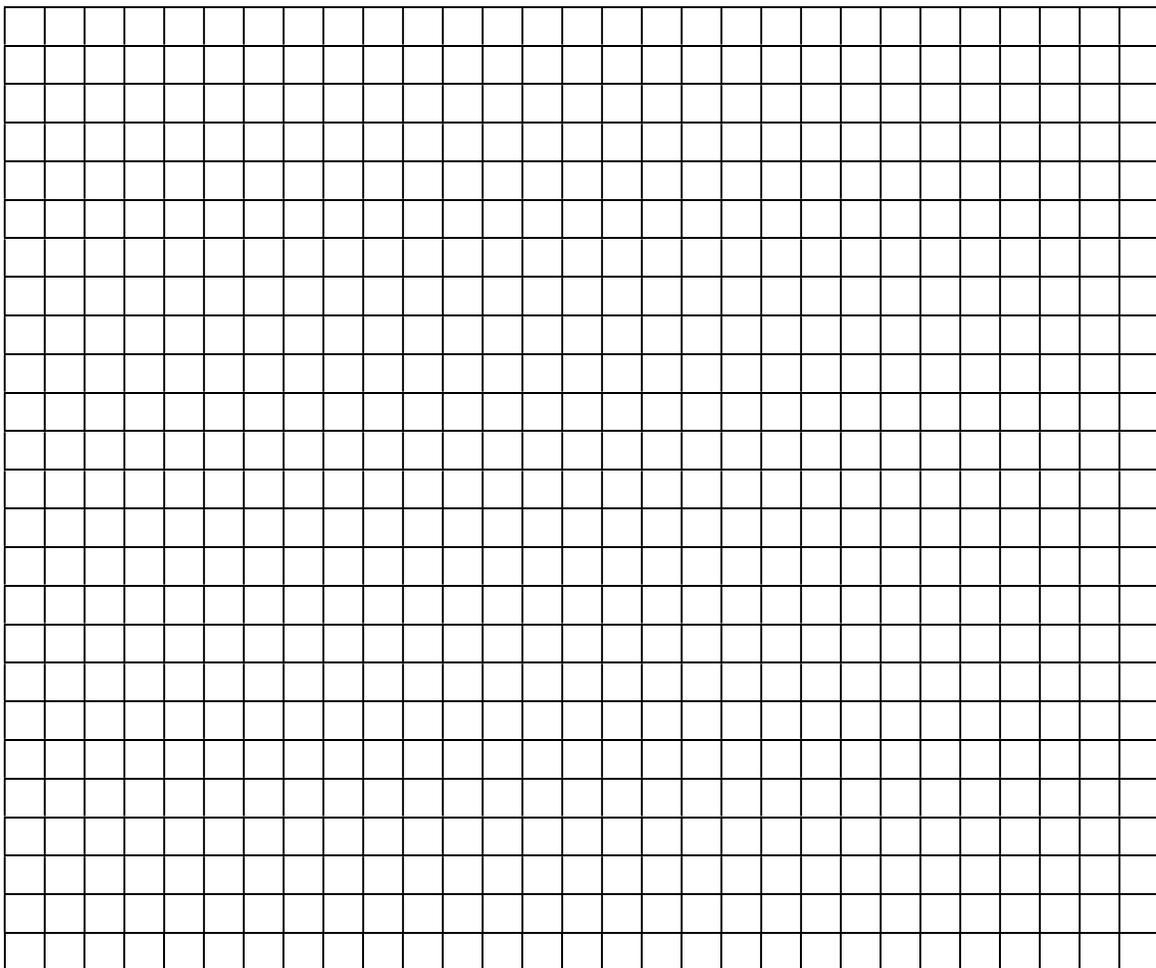
Odd: $f(-x) = \dots = -f(x)$ (Graph symmetric wrt origin: $(0,0)$)

Important Notes:

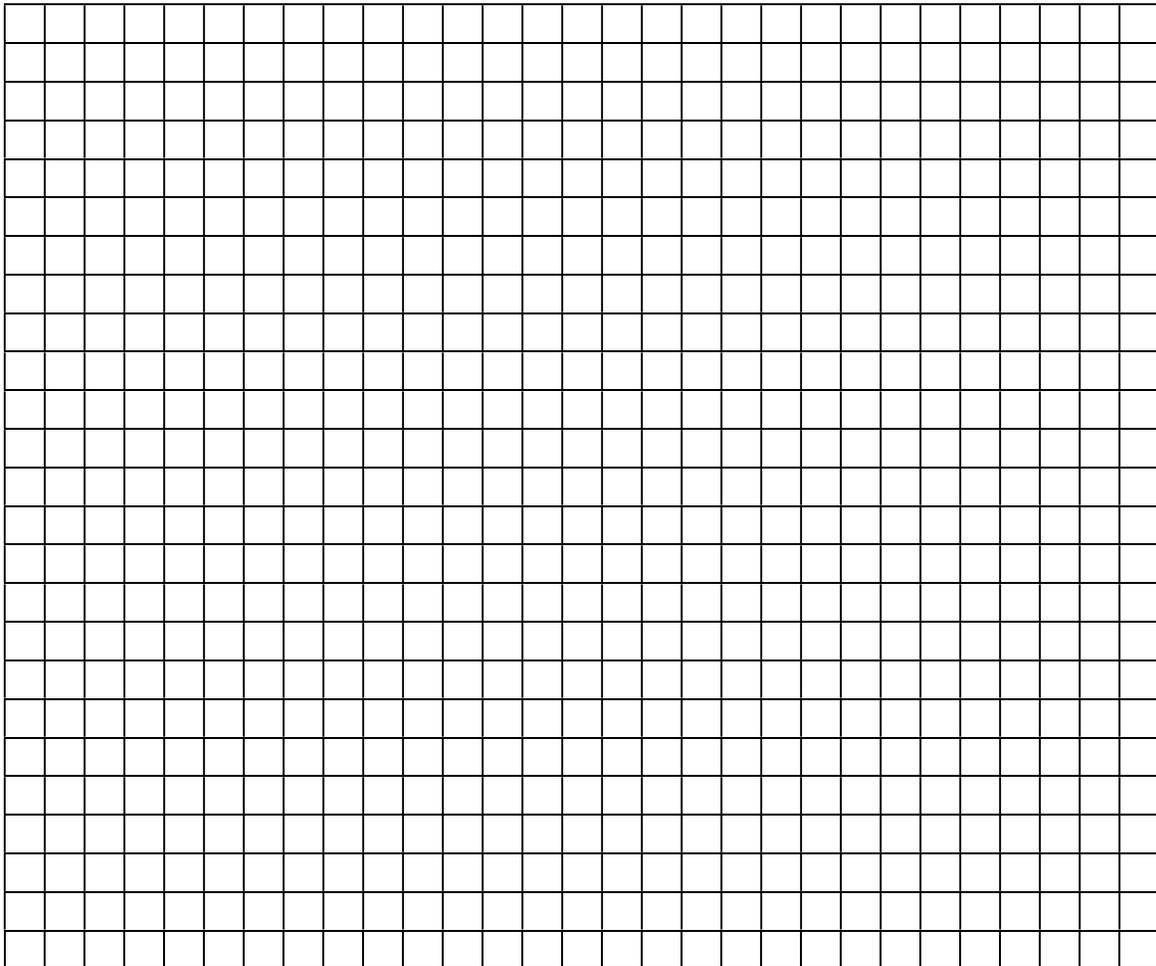
1. $\frac{\#}{\text{"BIG"}} = \text{"SMALL"}$

2. $\frac{\#}{\text{"SMALL"}} = \text{"BIG"}$

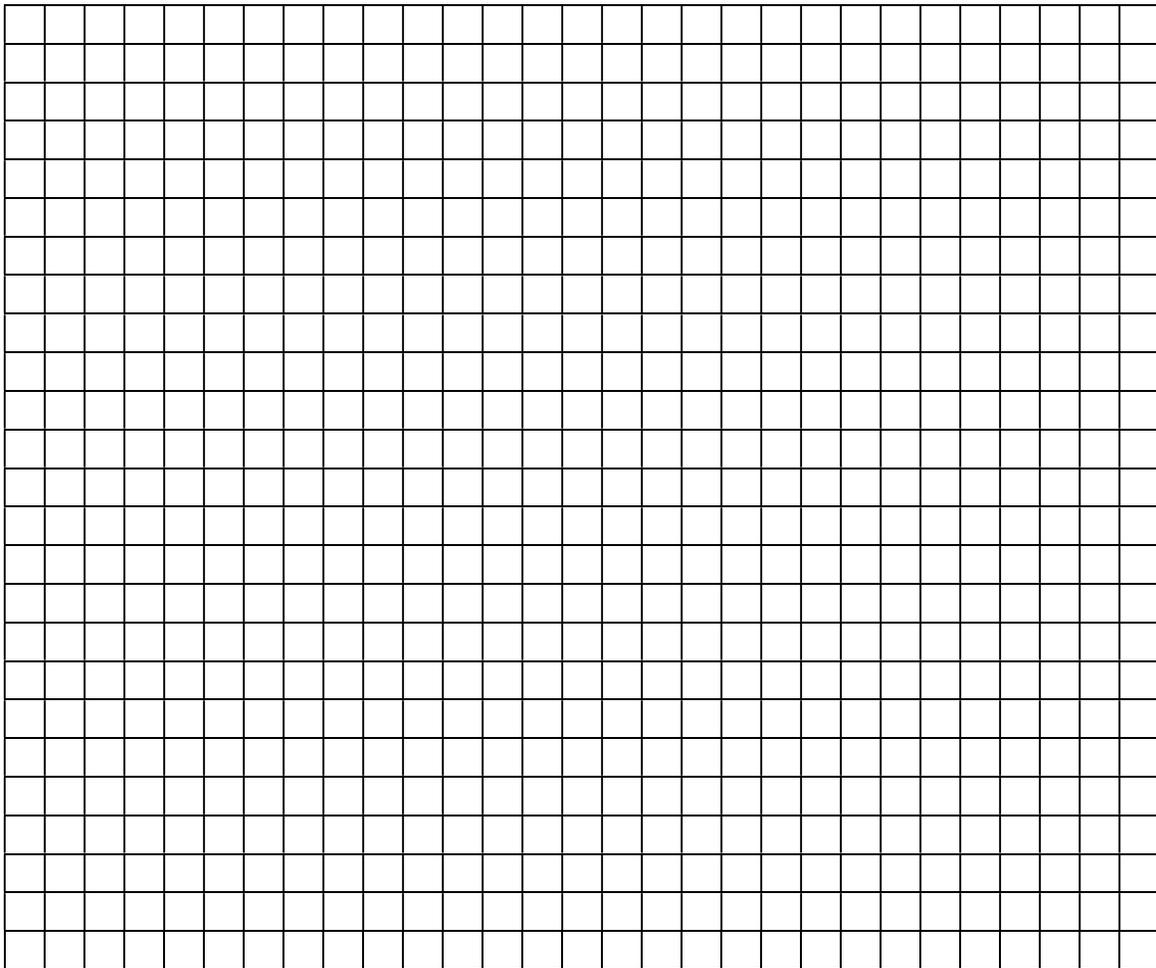
Example 01: $f(x) = \frac{2x^2}{x^2 + 4}$



Example 02: $f(x) = \frac{x+2}{x^3-4x}$



Example 03: $f(x) = \frac{x^2 + 4}{x}$



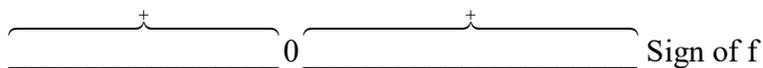
Example 01:

Find the first five (5) FUNCTION Summary Properties of $f(x) = \frac{2x^2}{x^2 + 4}$ and sketch its graph (best we can without the calculus)

Solution:

1. **Domain:** $\text{Dom } f = \mathbb{R}_x$
2. **Intercepts:** $(0, 0)$ is an x and y intercept POINT
3. **Continuity:**

$\text{Cont } f = \text{Dom } f = \mathbb{R}_x$



$\text{Pos } f = (-\infty, 0)_x \cup (0, +\infty)_x$

$\text{Neg } f = \phi$; Empty set

4. **Behavior at/toward Infinity:**

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{2x^2}{x^2 + 4} = \lim_{x \rightarrow -\infty} \frac{2}{1 + \frac{4}{x^2}} = 2$$

(Divide numerator & denominator by highest power: x^2)

Note: $\left(\frac{4}{\text{Big}} \rightarrow 0 \right); \left(\frac{\text{Num} = 2}{\text{Den} \approx 1} \rightarrow 2 \right)$

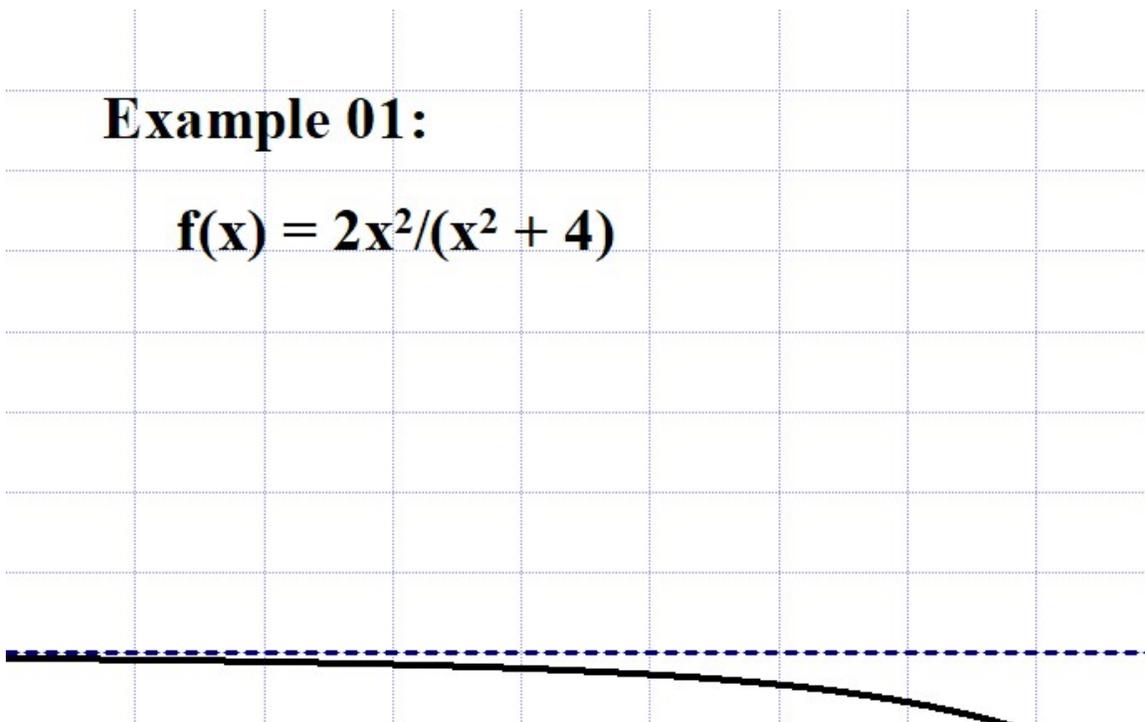
$\therefore y = 2$ is a horizontal asymptote

$\lim_{x \rightarrow +\infty} f(x) = 2$

5. **Symmetry:**

$$f(-x) = \frac{2(-x)^2}{(-x)^2 + 4} = \frac{2x^2}{x^2 + 4} = f(x) \Rightarrow \text{Even (Graph symmetric wrt } x = 0)$$

6. Graph what the information above yields and then look at the actual graph on the next page.



Example 2:

Find the first five (5) FUNction Summary Properties of $f(x) = \frac{x+2}{x^3-4x}$ and sketch its graph (best we can without the calculus)

Solution:1. **Domain:**

$$x^3 - 4x = 0 \Rightarrow x = -2, 0, 2$$

$$\text{Dom } f = \mathbb{R}_x \setminus \{-2, 0, 2\}_x$$

$$\text{Note: } f(x) = \frac{x+2}{x^3-4x} = \frac{x+2}{x(x-2)(x+2)} \stackrel{x \neq -2}{=} \frac{1}{x(x-2)}$$

2. **Intercepts:** There are NO y or x intercept POINTS since $-2 \notin \text{Dom } f$

3. **Continuity:**

$$\text{Cont } f = \text{Dom } f = \mathbb{R}_x \setminus \{-2, 0, 2\}_x$$

$$\overbrace{\quad}^{+} \quad -2 \quad \overbrace{\quad}^{+} \quad 0 \quad \overbrace{\quad}^{-} \quad 2 \quad \overbrace{\quad}^{+} \quad \text{Sign of } f$$

$$\text{Pos } f = (-\infty, -2)_x \cup (-2, 0)_x \cup (2, +\infty)_x$$

$$\text{Neg } f = (0, 2)_x$$

Discontinuity : $x = -2$:

Since $-2 \notin \text{Dom } f$, we try to find out what is happening to the $f(x)$ values for x values "close to" but unequal to -2 .

$$\text{From the left: } x \rightarrow -2^- \Rightarrow f(x) \rightarrow \frac{1}{8} \left(f(x) = \frac{1}{x(x-2)} \underset{x \rightarrow -2^-}{\Rightarrow} \frac{=1}{\approx 8} \right)$$

$$\text{Also written } \lim_{x \rightarrow -2^-} f(x) = \frac{1}{8}$$

$$\text{From the right: } x \rightarrow -2^+ \Rightarrow f(x) \rightarrow \frac{1}{8} \left(f(x) = \frac{1}{x(x-2)} \underset{x \rightarrow -2^+}{\Rightarrow} \frac{=1}{\approx 8} \right)$$

$$\text{Also written } \lim_{x \rightarrow -2^+} f(x) = \frac{1}{8}$$

$$\therefore \text{Hole: } \left(-2, \frac{1}{8} \right)$$

Discontinuity : $x = 0$:

Since $0 \notin \text{Dom } f$, we try to find out what is happening to the $f(x)$ values for x values "close to" but unequal to 0 .

$$\text{From the left: } x \rightarrow 0^- \Rightarrow f(x) \rightarrow +\infty \left(f(x) = \frac{1}{x(x-2)} \underset{x \rightarrow 0^-}{\Rightarrow} \frac{=1}{\approx 0(+)} \right)$$

$$\text{Also written } \lim_{x \rightarrow 0^-} f(x) = +\infty$$

\therefore Vertical Asymptote: $x = 0$

$$\text{From the right: } x \rightarrow 0^+ \Rightarrow f(x) \rightarrow -\infty \left(f(x) = \frac{1}{x(x-2)} \underset{x \rightarrow 0^+}{\Rightarrow} \frac{=1}{\approx 0(-)} \right)$$

$$\text{Also written } \lim_{x \rightarrow 0^+} f(x) = -\infty$$

Discontinuity : $x = 2$:

Since $2 \notin \text{Dom } f$, we try to find out what is happening to the $f(x)$ values for x values "close to" but unequal to 2.

$$\text{From the left: } x \rightarrow 2^- \Rightarrow f(x) \rightarrow -\infty \left(f(x) = \frac{1}{x(x-2)} \xrightarrow{x \rightarrow 2^-} \frac{=1}{\approx 0(-)} \right)$$

$$\text{Also written } \lim_{x \rightarrow 2^-} f(x) = -\infty$$

\therefore Vertical Asymptote: $x = 2$

$$\text{From the right: } x \rightarrow 2^+ \Rightarrow f(x) \rightarrow +\infty \left(f(x) = \frac{1}{x(x-2)} \xrightarrow{x \rightarrow 2^+} \frac{=1}{\approx 0(+)} \right)$$

$$\text{Also written } \lim_{x \rightarrow 2^+} f(x) = +\infty$$

4. Behavior at/toward Infinity:

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x+2}{x^3-4x} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x^2} + \frac{2}{x^3}}{1 - \frac{4}{x^2}} = 0$$

(Divide numerator & denominator by highest power: x^3)

$$\text{Note: } \left(\frac{1}{x^2}, \frac{2}{x^3}, \frac{4}{x^2} \Rightarrow \frac{= \#}{\text{Big}} \rightarrow 0 \right); \left(\frac{\text{Num} \approx 0}{\text{Den} \approx 1} \rightarrow 0 \right)$$

$\therefore y = 0$ is a horizontal asymptote

$$\lim_{x \rightarrow +\infty} f(x) = 0$$

5. Symmetry:

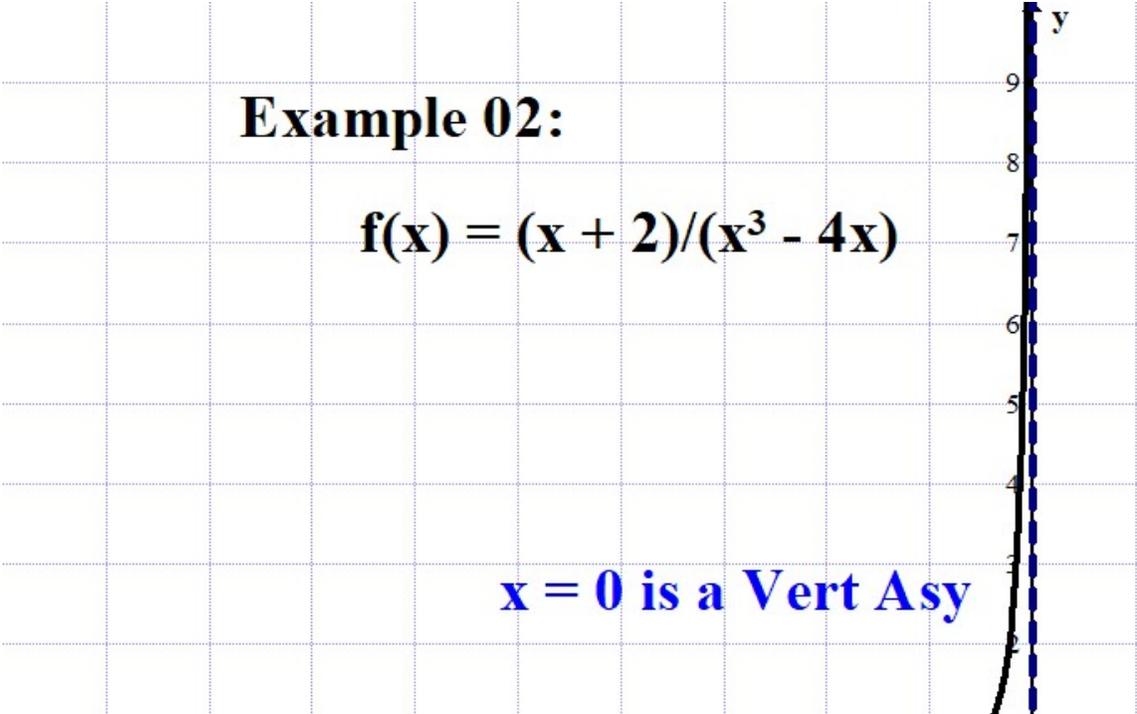
$$f(-x) = \frac{(-x)+2}{(-x)^3-4(-x)} = \frac{x-2}{x^3-4x} \neq \begin{cases} f(x) \\ -f(x) \end{cases} \Rightarrow \text{Neither even/odd}$$

6. Graph what the information above yields and then look at the actual graph below:

Example 02:

$$f(x) = (x + 2)/(x^3 - 4x)$$

$x = 0$ is a Vert Asy



Example 3:

Find the first five (5) FUNction Summary Properties of $f(x) = \frac{x^2 - 4}{x^2 - 6x - 27}$ and sketch its graph (best we can without the calculus)

Solution:

1. **Domain:**

$$x^2 - 6x - 27 \stackrel{\text{SET}}{=} 0 \Rightarrow (x+3)(x-9) = 0 \Rightarrow x = -3, 9$$

$$\text{Dom } f = \mathbb{R}_x \setminus \{-3, 9\}_x$$

2. **Intercepts:**

y-intercept POINT: $\left(0, \frac{4}{27}\right)$; Set $x = 0$; Evaluate

x-intercept POINT:

$(-2, 0)$; $(+2, 0)$; Set $f(x) = x^2 - 4 = (x+2)(x-2) \stackrel{\text{SET}}{=} 0$; Solve

3. **Continuity:**

Cont $f = \text{Dom } f = \mathbb{R}_x \setminus \{-3, 9\}_x$; same as domain for rational functions

$$\text{Note: } f(x) = \frac{x^2 - 4}{x^2 - 6x - 27} = \frac{(x-2)(x+2)}{(x-9)(x+3)}$$

$\overbrace{\quad}^{+} \quad -3 \quad \overbrace{\quad}^{-} \quad -2 \quad \overbrace{\quad}^{+} \quad 2 \quad \overbrace{\quad}^{-} \quad 9 \quad \overbrace{\quad}^{+}$ Sign of f

$$\text{Pos } f = (-\infty, -3)_x \cup (-3, 2)_x \cup (9, +\infty)_x$$

$$\text{Neg } f = (-3, -2)_x \cup (2, 9)_x$$

Discontinuity : $x = -3$:

Since $-3 \notin \text{Dom } f$, we try to find out what is happening to the $f(x)$

values for x values "close to" but unequal to -3 .

$$\text{From the left: } x \rightarrow -3^- \Rightarrow f(x) \rightarrow +\infty \left(f(x) = \frac{(x-2)(x+2)}{(x+3)(x-9)} \underset{x \rightarrow -3^-}{\Rightarrow} \frac{\approx 5}{\approx 0(+)} \right)$$

$$\text{Also written } \lim_{x \rightarrow -3^-} f(x) = +\infty$$

\therefore Vertical Asymptote: $x = -3$

$$\text{From the right: } x \rightarrow -3^+ \Rightarrow f(x) \rightarrow -\infty \left(f(x) \underset{x \rightarrow -3^+}{\Rightarrow} \frac{\approx 5}{\approx 0(-)} \right)$$

$$\text{Also written } \lim_{x \rightarrow -3^+} f(x) = -\infty$$

Discontinuity : $x = 9$:

Since $9 \notin \text{Dom } f$, we try to find out what is happening to the $f(x)$ values for x values "close to" but unequal to 9.

$$\text{From the left: } x \rightarrow 9^- \Rightarrow f(x) \rightarrow -\infty \left(f(x) = \frac{(x-2)(x+2)}{(x+3)(x-9)} \xrightarrow{x \rightarrow 9^-} \frac{\approx 77}{\approx 0(-)} \right)$$

$$\text{Also written } \lim_{x \rightarrow 9^-} f(x) = -\infty$$

Vertical Asymptote: $x = 9$

$$\text{From the right: } x \rightarrow 9^+ \Rightarrow f(x) \rightarrow +\infty \left(f(x) \xrightarrow{x \rightarrow 9^+} \frac{\approx 77}{\approx 0(+)} \right)$$

$$\text{Also written } \lim_{x \rightarrow 9^+} f(x) = +\infty$$

4. Behavior at/toward Infinity:

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x^2 - 4}{x^2 - 6x - 27} = \lim_{x \rightarrow -\infty} \frac{1 - \frac{4}{x^2}}{1 - \frac{6}{x} - \frac{27}{x^2}} = 1$$

(Divide numerator & denominator by highest power: x^2)

$$\text{Note: } \left(\frac{4}{x^2}, \frac{6}{x}, \frac{27}{x^3} \Rightarrow \frac{\#}{\text{Big}} \rightarrow 0 \right); \left(\frac{\text{Num} \rightarrow 1}{\text{Den} \rightarrow 1} \rightarrow 1 \right)$$

$\therefore y = 1$ is a horizontal asymptote

$$\lim_{x \rightarrow +\infty} f(x) = 1$$

If there is an x -value such that $f(x) = 1$, then the graph and Horizontal Asymptote intersect:

$$f(x) \stackrel{\text{SET}}{=} 1 \Rightarrow \frac{x^2 - 4}{x^2 - 6x - 27} = 1$$

$$\Rightarrow x^2 - 4 = x^2 - 6x - 27$$

$$\Rightarrow -4 = -6x - 27$$

$$\Rightarrow 6x = -27 + 4 = -23$$

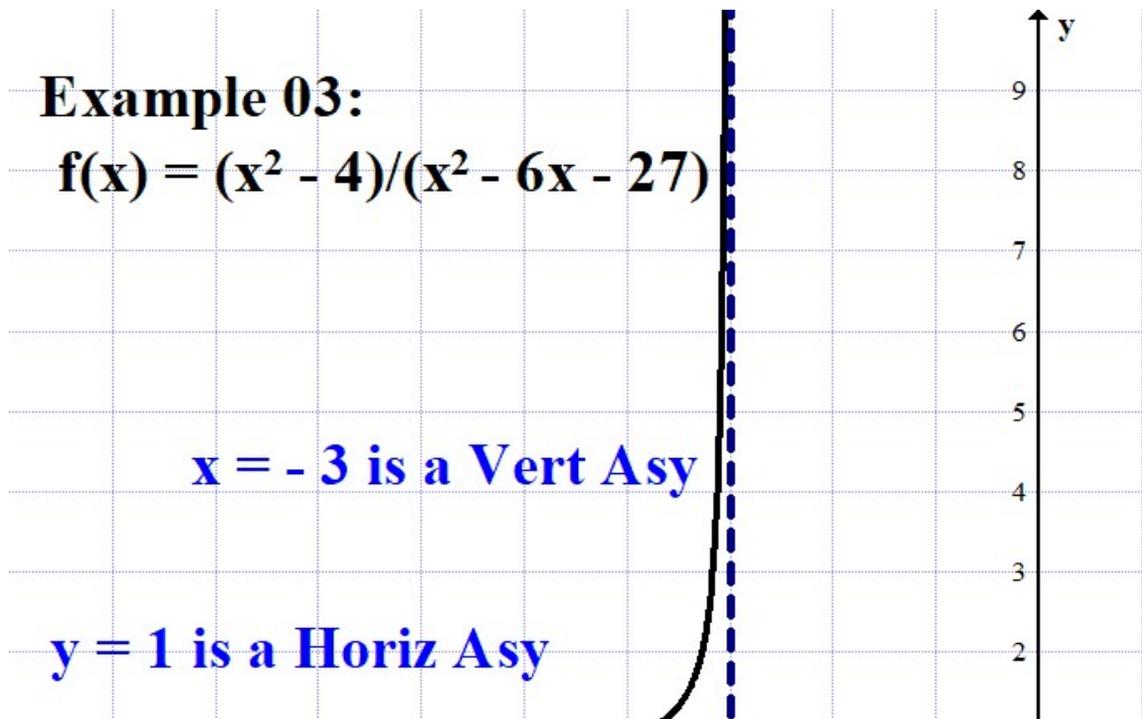
$$\Rightarrow x = -\frac{23}{6} = -3\frac{5}{6}$$

$$\Rightarrow \text{Graph intersects Horiz Asy at } \left(-\frac{23}{6}, 1 \right)$$

5. Symmetry:

$$f(-x) = \frac{(-x)^2 - 4}{(-x)^2 - 6(-x) - 27} = \frac{x^2 - 4}{x^2 + 6x - 27} \neq \begin{cases} f(x) \\ -f(x) \end{cases} \Rightarrow \text{Neither even/odd}$$

6. Graph what the information above yields and then look at the actual graph on the next page.



Example 4:

Find the first five (5) FUNction Summary Properties of $f(x) = \frac{x^2 - 1}{x - 1}$ and sketch its graph (best we can without the calculus)

Solution:

1. **Domain:**

$$\text{Dom } f = \mathbb{R}_x \setminus \{1\}_x$$

$$f(x) = \frac{x^2 - 1}{x - 1} = x + 1 \quad x \neq 1$$

2. **Intercepts:**

y-intercept POINT: (0,1)

x-intercept POINT: (-1,0) since $1 \notin \text{Dom } f$

3. **Continuity:**

$$\text{Cont } f = \text{Dom } f = \mathbb{R}_x \setminus \{1\}_x$$

$$\overbrace{\quad}^{-} -1 \overbrace{\quad}^{+} 1 \overbrace{\quad}^{+} \text{ Sign of } f$$

$$\text{Pos } f = (-1, 1)_x \cup (1, +\infty)_x$$

$$\text{Neg } f = (-\infty, -1)_x$$

Discontinuity : $x = 1$

Since $1 \notin \text{Dom } f$, we try to find out what is happening to the $f(x)$ values for x values "close to" but unequal to 1.

$$\text{From the left: } x \rightarrow 1^- \Rightarrow f(x) \rightarrow 2 \left(f(x) = x + 1 \Rightarrow \underset{x \rightarrow 1^-}{\approx} 2 \right)$$

$$\text{Also written } \lim_{x \rightarrow 1^-} f(x) = 2$$

$$\text{From the right: } x \rightarrow 1^+ \Rightarrow f(x) \rightarrow 2 \left(f(x) \underset{x \rightarrow 1^+}{\approx} 2 \right)$$

$$\text{Also written } \lim_{x \rightarrow 1^+} f(x) = 2$$

\therefore Hole: (1,2)

4. **Behavior at/toward Infinity:**

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow -\infty} \frac{1 - \frac{1}{x^2}}{\frac{1}{x} - \frac{1}{x^2}} = -\infty$$

(Divide numerator & denominator by highest power: x^2)

$$\text{Note: } \left(\frac{1}{x^2}, \frac{1}{x} \Rightarrow \frac{=1}{\approx \text{Big}} \rightarrow 0 \right); \left(\frac{\text{Numer} \rightarrow 1}{\text{Denom} \rightarrow 0(-)} \rightarrow -\infty \right)$$

$$\lim_{x \rightarrow +\infty} f(x) = +\infty$$

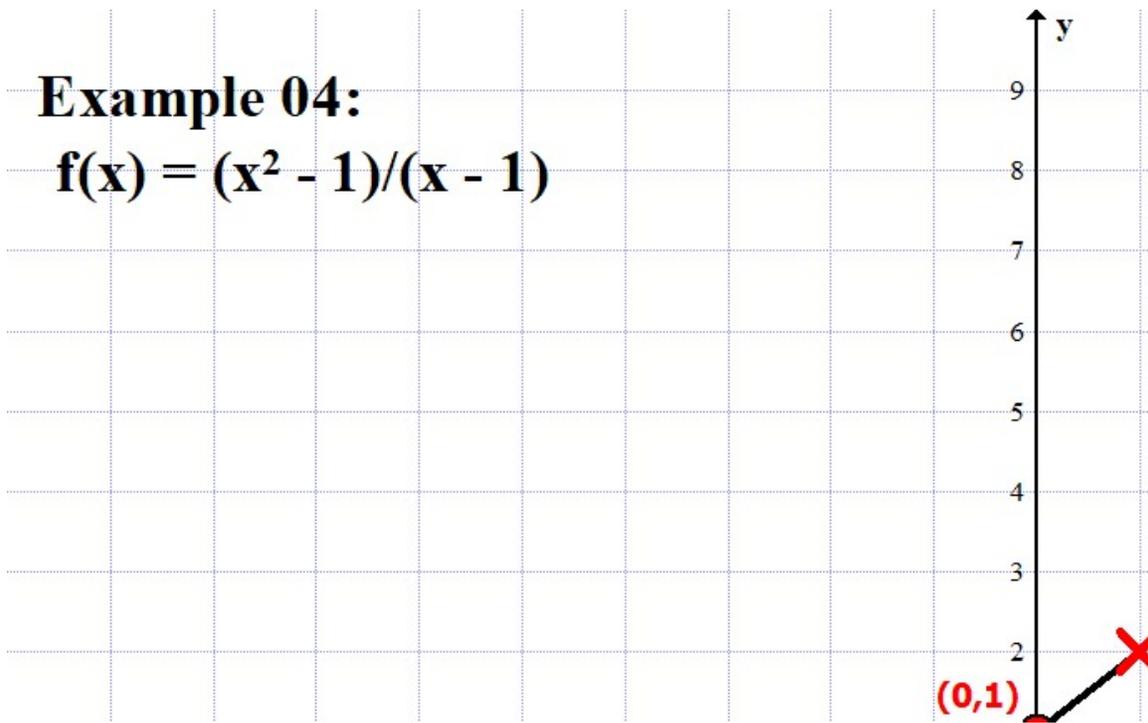
5. **Symmetry:**

$$f(-x) = \frac{(-x)^2 - 1}{(-x) - 1} = -\frac{x^2 - 1}{x + 1} \neq \begin{cases} f(x) \\ -f(x) \end{cases} \Rightarrow \text{Neither even/odd}$$

6. Graph what the information above yields and then look at the actual graph on the next page.

Example 04:

$$f(x) = (x^2 - 1)/(x - 1)$$



Example 5:

Find the first five (5) FUNction Summary Properties of $f(x) = \frac{x^2 + 4}{x}$ and sketch its graph (best we can without the calculus)

Solution:

1. **Domain:**

$$\text{Dom } f = \mathbb{R}_x \setminus \{0\}_x$$

2. **Intercepts:**

There are NO y or x intercept POINTS

3. **Continuity:**

$$\text{Cont } f = \text{Dom } f = \mathbb{R}_x \setminus \{0\}_x$$

$$\overbrace{\hspace{10em}}^{-} \quad 0 \quad \overbrace{\hspace{10em}}^{+} \quad \text{Sign of } f$$

$$\text{Pos } f = (0, +\infty)_x$$

$$\text{Neg } f = (-\infty, 0)_x$$

$$\text{Discontinuity : } x = 0$$

Since $0 \notin \text{Dom } f$, we try to find out what is happening to the $f(x)$ values for x values "close to" but unequal to 0.

$$\text{From the left: } x \rightarrow 0^- \Rightarrow f(x) \rightarrow -\infty \left(f(x) = \frac{x^2 + 4}{x} \Rightarrow \frac{\approx 4}{\approx 0(-)} \right)$$

$$\text{Also written } \lim_{x \rightarrow 0^-} f(x) = -\infty$$

\therefore Vertical Asymptote: $x = 0$

$$\text{From the right: } x \rightarrow 0^+ \Rightarrow f(x) \rightarrow +\infty \left(f(x) \Rightarrow \frac{\approx 4}{\approx 0(+)} \right)$$

$$\text{Also written: } \lim_{x \rightarrow 0^+} f(x) = +\infty$$

4. **Behavior at/toward Infinity:**

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x^2 + 4}{x} = \lim_{x \rightarrow -\infty} \frac{1 + \frac{4}{x^2}}{\frac{1}{x}} = -\infty$$

(Divide numerator & denominator by highest power: x^2)

$$\text{Note: } \left(\frac{4}{x^2}, \frac{1}{x} \Rightarrow \frac{\#}{\text{Big}} \rightarrow 0 \right); \left(\frac{\text{Numer} \rightarrow 1}{\text{Denom} \rightarrow 0(-)} \rightarrow -\infty \right)$$

$$\lim_{x \rightarrow +\infty} f(x) = +\infty$$

$$f(x) = \frac{x^2 + 4}{x} = x + \frac{4}{x} \approx x \text{ when } x \text{ is BIG!}$$

$\Rightarrow y = x$ is a Slant Asymptote

5. **Symmetry:**

$$f(-x) = \frac{(-x)^2 + 4}{(-x)} = -\frac{x^2 + 4}{x} = -f(x) \Rightarrow \text{Odd (Graph symmetric wrt (0,0))}$$

6. Graph what the information above yields and then look at the actual graph on the next page.

Example 05:

